Separating markets

In 1849, the French economist Dupuit described a model of monopolistic competition in an article published in the amusingly-titled *Journal of Bridges and Roads*. One of his central claims is that a bridge is, in effect, a monopoly: people wishing to cross the river (in this case, the Seine) either pay a toll to use the bridge or don't cross the river. Of course, there are issues regarding the fact that there are other bridges, ferries, etc. but back in the mid-1800s is was enough of a pain to get from bridge to bridge that citizens either paid and crossed on their local bridge or didn't cross at all. This is a fair analogy to a monopoly: a single firm controls all access to a particular good and you are either willing to pay their fee for access to the good, or you are not.

He begins by relating the particular example of the Pont d'Arcole, a bridge in central Paris. At some point prior to his paper, the toll of the bridge was reduced from 5 francs to 0 francs; traffic increased tenfold. He posited the following demand structure for bridge use, in terms of total crossings at a particular price.

P	0	1	2	3	4	5	6	7	8
Q	100	55	37	24	15	10	5	2	0

Figure 1: demand for bridge crossings, as a function of the toll charged

If we ignore the cost of bridge maintenance, running a bridge is pure upside. By posting a toll of 5 francs, the bridge owner makes a profit of $5(10) = 50$ francs; by posting a toll of 0 francs, the bridge owner makes a profit of 0 francs. So if these are the two choices, the bridge owner will clearly prefer posting a toll of 5 francs to allowing open access for free.

However, the bridge owner can do better. Computing the monopolist's optimum price is by now a familiar exercise for us, but it's useful to do it in a discrete space (we're assuming here that prices such as 2.5 francs are disallowed, even though francs were divisible into centimes). According to our assumption of 0 cost of operation, the bridge owner's profits as a function of price are

P	0	1	2	3	4	5	6	7	8
π	0	55	74	72	60	50	30	14	0

Figure 2: profit to the bridge owner, as a function of the toll charged

Looking down the table, we see that the bridge owner is best off by charging a toll of 2 francs. So by intelligently pricing a bridge crossing at 2 francs, the bridge owner is better off by $74 - 50 = 24$ francs and $55 - 10 = 45$ additional crossings are achieved. Everyone is better off compared to the old system (although not compared to the free-crossing regime)!

This is better than the former status quo, but it's possible everyone can be made better off still. Dupuit notes, roughly, that French society at the time is comprised of three social classes: peasants, laborers, and the bourgeoisie. Peasants are poor, and can afford to cross the bridge only if doing so costs nothing. Laborers cross the bridge to get to work, and value crossing at 1 or 2 francs. The bourgeoisie cross at their leisure; being rich, they don't care about the toll of the bridge (up to a point) and all value bridge use at at least 3 francs.

Is it possible that the bridge owner can use this social stratification to improve his surplus? In the mid-1800s, laborers were the only social class which wore short jackets (apparently the upper class wore long coats, while the peasants froze). Then maybe the bridge owner can discriminate based on the type of jacket or coat the potential crosser is wearing. Dupuit is quick to mention, however, that price discrimination on the basis of clothing will be completely ineffective: nothing is to prevent the bourgeoisie from swapping their coats for jackets before they cross the bridge, then "classing up" once they've crossed! Clearly a better method is needed.

There are quite a few methods of discriminating between French social classes in the 19th century, but Dupuit notes an effective method for splitting markets without discriminating based on appearance, name, ID, or any immediately-observable features. Peasants are willing to cross the bridge at any time of day, if the price is right. Laborers will cross only during "rush hour" (in our modern parlance) as they travel to work. The bourgeoisie really enjoy sleeping in, and avoid travelling at rush hour at all costs.

This gives rise to a natural method of separating markets: charge one toll during rush hour, and another outside of rush hour. What is demand in these two (separate) parts of the day? Recall that we've got a correspondence between when people are willing to cross and what price they are willing to pay: peasants and laborers value crossing at 2 francs or less and are willing to cross during rush hour. Peasants and the bourgeoisie value crossing at 0 francs or above 3 francs and are willing to cross outside of rush hour. Demand will then look like

P	0	1	2
Q	76	31	13

Figure 3: demand for bridge crossings during rush hour

How did we obtain these demand figures? Using the original demand table, we see that $100 - 55 = 45$ potential bridge crossers value crossing at 0 francs (that is, there are 45 peasants). Similarly, $55 - 37 = 18$ people value crossing at 1 franc and $37 - 24 = 13$ people value crossing at 2 francs. We can then construct the rush hour demand as 13 people valuing crossing at at least 2 francs, $13 + 18 = 31$ people valing crossing at at least 1 franc, and $31 + 45 = 76$ people valuing crossing at at least 0 francs.

What is the monopolist's optimal price during rush hour? We compute profits as in Figure 4. So the bridge owner will optimize by posting a price of 1 franc during rush hour.

What happens outside of rush hour? We perform similar calculations to those during rush hour and obtain

P	0	1	2
π	0	31	26

Figure 4: profit from bridge crossings during rush hour

a demand curve of

P	0	1	2	3	4	5	6	7	8
Q	69	24	24	24	15	10	5	2	0

Figure 5: demand for bridge crossings outside of rush hour

These demand figures were obtained via a method identical to that above. Profits outside of rush hour, as a function of price charged are then

P	0	1	2	3	4	5	6	7	8
π	0	24	48	72	60	50	30	14	0

Figure 6: demand for bridge crossings, as a function of the toll charged

The bridge owner will then charge a toll of 3 outside of rush hour.

As compared to our usual "separation of markets" routine, there is some demand — namely, that of peasants — which occurs in both markets. We had the option of splitting this to one market or another, but it does not affect the outcome of the question; we can think of some very small group of peasants crossing the bridge as frequently as possible, since they ostensibly have nothing better to do.

So by splitting markets, the bridge owner has moved from charging a toll of 2 francs and making a profit of 74 francs to charging a toll of 1 franc during rush hour and a toll of 3 francs at all other times. His profits are then $72 + 31 = 103$ francs, and the bridge owner is therefore significantly better off.

What about consumers? If a price of 2 francs is charged to the entire market, consumer surplus is

$$CS = 5(2) + 4(3) + 3(5) + 2(5) + 1(9) = 56$$

That is, the 2 consumers who value crossing at 7 francs obtain a surplus of 5 francs above the going price of 2 francs; the 3 consumers who value crossing at 6 francs obtain a surplus of 4 francs above the going price of 2 francs, etc.

In the separate market case, consumer surplus is divided between rush hour and not-rush hour. We compute the separate consumer surpluses as

$$CS_{\text{rush}} = 1(13)$$
$$CS_{\text{regular}} = 4(2) + 3(3) + 2(5) + 1(5)$$

Total consumer surplus is then

$$CS = CS_{\text{rush}} + CS_{\text{regular}} = 45$$

Total consumer surplus is then less than in the single-market case, and consumers are worse off.

That the firm is at least as well off is a general feature of separating markets: consider the fact that the firm still has the option of charging the same price in each market, so should never do worse than in a single market. However, that consumers are worse of is *not* a general feature of separating markets. There are certainly occasions in which everyone winds up better off by splitting markets into separate pieces. For an example of such a market, feel free to repeat the above exercise with the following numbers (which are distinct from Dupuit's):

P	0	1	2	3	4	5	6
Q	100	70	50	35	20	10	0

Figure 7: possible new demand curve for bridge crossings

The Costco effect

While we are familiar with firms facing a fixed cost for production, what would happen if consumers were charged a fixed cost for consumption? Although this sounds odd, this is not unlike the mechanism we see at stores like Costco and Sam's Club: consumers are charged a fixed cost for access to the store and are then charged a per-item fee for each TV they buy.

In order to make sense of the ensuing optimization, we need to reconsider what a demand curve really is: it is an aggregation of individual consumer demands. In a sense, market demand is what happens when firms cannot separate consumers into individual markets in which they could extract as much surplus as they liked.

For the sake of exposition, let's assume for now that there is just one type of consumer in the economy; this consumer has a demand function $Q_D = 1 - P$. There is a monopolist who faces a marginal cost of 0 for production. When we solve the traditional monopolist's problem,

$$\max_P (1 - P) P$$

we obtain a price level of $\frac{1}{2}$ and a quantity produced of $\frac{1}{2}$.

At this level of production, total surplus in the economy is

$$\int_0^{\frac{1}{2}} 1 - q dq = \frac{3}{8}$$

Since the firm makes a profit of $PQ = \frac{1}{4}$, consumer surplus is $\frac{1}{8}$.

In a sense, this is money that the firm is leaving on the table; since the consumer is better off from entering the economy — by $\frac{1}{8}$ — it may be possible for the firm to extract some more of the consumer's surplus and be better off. We introduce the concept of the *two-part tariff*. In this system, the firm charges some amount to the consumer in order to have the right to purchase the good and then some additional amount to purchase the good itself. Here, since the consumer obtains a surplus of $\frac{1}{8}$, the firm can charge up to $\frac{1}{8}$ for the consumer to have the privilege of entering the market; this will enter positively into profits!

More formally, there is some market price P along with a fixed cost T (to the consumer) to enter the market. The firm's problem becomes

$$\max_P PQ_D(P) + T - C(Q_D(P))$$

With one consumer, the problem isn't too rough: the firm will extract all available consumer surplus. We know that consumer surplus is total surplus less firm profits, or

$$CS = \int_0^{Q_D(P)} P(Q)\, dQ - PQ_D(P)$$

Substituting in, the firm's problem becomes

$$\max_P \int_0^{Q_D(P)} P(Q)\,dQ - C(Q_D(P))$$

From first-order conditions, this yields (at the optimum)

$$Q_D'(P)P(Q_D(P)) = C'(Q_D(P))\,Q_D'(P) \qquad \Longrightarrow \qquad P(Q) = C'(Q)$$

In the particular setup above, $P(Q) = 1 - Q$ and $C'(Q) = MC = 0$. This tells us that $Q = 1$.

When $Q = 1$, consumer surplus is

$$CS = \int_0^1 1 - Q dQ - 0(1) = \frac{1}{2}$$

Then the firm can charge a fixed tariff of $T = \frac{1}{2}$ without forcing the consumer to remain out of the market. In this case, the optimal two-part tariff is

$$T = \frac{1}{2}, P = 0$$

with a quantity supplied to the market of $Q = 1$. The firm makes a profit of $\pi = \frac{1}{2}$, as compared to $\frac{1}{4}$ in the monopoly case (and $\frac{3}{8}$ from full extraction in the monopoly case). The consumer is unquestionably worse off, but the firm is more than happy.

Solving for the optimal two-part tariff in the case of one consumer is not terribly difficult, due to there being few strategic concerns and a great deal of algebraic cancellation. To see how strategic considerations enter the problem let's keep the same setup from above and add a second [type of] consumer; denote the existing consumer's demand by $Q_D^1 = 1 - P$ and let the new consumer's be $Q_D^2 = k - P$ for some $k > 1$.

To establish a baseline, let's consider the monopolist's problem *without* a two-part tariff. The firm has the option of facing the two consumers together in a conjoined market, or catering to just one of the consumers and pricing the other out. We can solve these as two distinct monopoly problems and then piece together the firm's optimal strategy.

Demand in the joint market is

$$Q_D = \begin{cases} k + 1 - 2P & \text{if } P < 1 \\ k - P & \text{if } 1 \le P \le k \\ 0 & \text{otherwise} \end{cases}$$

To solve the firm's problem, we'll assume a case-by-case structure. Obviously the firm will never set $P > k$ since it will earn 0 profits; so here we have only two cases:

- $P < 1$

  $$\max_P (k + 1 - 2P)P$$

  From first-order conditions, $P = \frac{k+1}{4}$. The firm's profits are then $\pi = \frac{1}{8}(k+1)^2$. Notice that this price only squares with the assumption that $P < 1$ when $k < 3$.

- $1 \le P < k$

  $$\max_P (k - P)P$$

  From first-order conditions, $P = \frac{k}{2}$. The firm's profits are then $\pi = \frac{k^2}{4}$. Notice that this price only squares with the assumption that $1 \le P < k$ when $k \ge 2$.

Which price will the firm choose to set? It will set a price according to the level which maximizes profits. We can check when it's advantageous to cut the customer of the first type out of the market by comparing profits when it does to profits when it does not.

$$
\begin{aligned}
& \frac{k^2}{4} \circ \frac{1}{8}(k+1)^2 \\
\iff\quad & 2k^2 \circ (k+1)^2 \\
\iff\quad & \sqrt{2}k \circ k+1 \\
\iff\quad & k \circ \frac{1}{\sqrt{2}-1}
\end{aligned}
$$

Then when $k \geq \frac{1}{\sqrt{2}-1}$ the firm chooses $P = \frac{k}{2}$ and cuts the "lower" consumer out of the market, and when $k < \frac{1}{\sqrt{2}-1}$ the firm chooses $P = \frac{k+1}{4}$ and captures the joint market of both types of consumer.

Now, what if the firm can post a two-part tariff? It still faces the decision of whether it should price to keep the low-type consumer out of the market (although it's not clear yet that this decision does not have an obvious solution, irrespective of k). Since this case is much simpler, let's look first at the case in which the firm sets a price/tariff high enough to keep the low-type consumer out of the market; then its tariff plan need only respect the high-type's demand curve. Then this setup is the same as a single-consumer market! We solved this above (for the consumer with $Q_D = 1 - P$); since nothing in the problem changes other than the intercept in the demand function, we can see that $P = 0$, $Q = k$, and $T = CS = \frac{k^2}{2}$. Then profits are $\pi = \frac{k^2}{2}$.

However, what happens if the firm prices so that both consumers enter the market? First, let's establish a little bit of base intuition: the consumer with the higher demand curve will always obtain a (weakly) higher level of surplus than the consumer with the lower demand curve. Since this is the case, the firm cannot extract all consumer surplus from the market, but it can still capture some portion. Intuitively, it can extract all of the surplus of the low-type consumer — leaving her indifferent to entering the market or not — through a tariff; but in doing so, it has left some of the high-type consumer's surplus on the table.

So we can see that if the firm allows both customers to enter the market, it sets a tariff so that the low-type customer extracts no surplus; denote this consumer's surplus (absent tariffs) by CS_1. Then the monopolist's problem is

$$\max_P (k+1-2P)P + 2T(P) = \max_P (k+1-2P)P + 2CS_1(P)$$

Notice that the firm obtains the tariff twice, once for each consumer!

Recalling that the low-type consumer's demand is $Q_D^1 = 1 - P$, we have

$$CS_1(P) = \int_0^{1-P} 1 - Q dQ - (1-P)P$$

That is, consumer surplus is the area under the demand curve less the profits the firm receives (quantity $1 - P$ at price P gives profits of $(1 - P)P$).

We may then restate the firm's maximization as

$$\max_P (k+1-2P)P + 2\left(\int_0^{1-P} 1 - Q dQ - (1-P)P\right) = \max_P (k-1)P + 2\int_0^{1-P} 1 - Q dQ$$

Taking first-order conditions with respect to P, we find

$$
\begin{aligned}
& 0 = k - 1 - 2\left(1 - (1-P)\right) \\
& = k - 1 - 2P \\
\iff\quad & P = \frac{k-1}{2}
\end{aligned}
$$

At this price level, the low-type consumer's surplus (absent tariffs) is

$$\begin{aligned}\int_0^{\frac{1-k-12}{1}} -QdQ - \left(1-\frac{k-1}{2}\right)\frac{k-1}{2} &= \int_0^{\frac{3-k}{2}} 1-QdQ - \left(\frac{3-k}{2}\right)\frac{k-1}{2}\\ &= \frac{3-k}{2} - \frac{1}{8}(3-k)^2 + \left(\frac{3-k}{2}\right)^2 - \frac{3-k}{2}\\ &= \frac{(3-k)^2}{8}\end{aligned}$$

This gives us an explicit expression for the optimal tariff T.

The firm's profits are then determined to be

$$\begin{aligned}\left(k+1-2\left(\frac{k-1}{2}\right)\right)\left(\frac{k-1}{2}\right)+2\left(\frac{(3-k)^2}{8}\right) &= 2\left(\frac{k-1}{2}\right)+\frac{1}{4}\left(9-6k+k^2\right)\\ &= \frac{1}{4}\left(5-2k+k^2\right)\end{aligned}$$

When is it optimal to set this two-part tariff and allow both consumers to enter, versus the tariff structure which keeps the low-type consumer out of the market? Again, we compare profits.

$$\begin{aligned}& \frac{k^2}{2} \circ \frac{1}{4}\left(5-2k+k^2\right)\\ \iff\quad & 2k^2 \circ 5-2k+k^2\\ \iff\quad & k^2+2k-5 \circ 0\end{aligned}$$

We can solve for the threshold k at which the firm is indifferent between letting the low-type consumer enter and forcing her out.

$$k = \frac{-2 \pm \sqrt{24}}{2}$$

Clearly only the + solution will be valid, so we have

$$k = -1+\sqrt{6}$$

Then when $k > \sqrt{6}-1$, the firm will keep the low-type consumer out of the market. We can tabulate these results as

	$k < \sqrt{6}-1$	$\sqrt{6}-1 \leq k$
P	$\left(\frac{k-1}{2}\right)$	0
T	$\left(\frac{(3-k)^2}{8}\right)$	$\left(\frac{k^2}{2}\right)$
Q	2	k
π	$\frac{1}{4}\left(5-2k+k^2\right)$	$\left(\frac{k^2}{2}\right)$

As an exercise, compare these profits to the monopoly profits achievable without a tariff; which are greater? What is the intuition?

It is worth noting that we carried out this entire exercise without considering the costs of the firm. For the purposes of leaving the method of determining the optimal strategy, it is nice to not have to worry about nontrivial cost functions. On the whole, the analysis would not change at all; we would simply be subject to altered first-order conditions.

Nonstandard price competition

Version 1

Suppose there are two firms competing in prices in the standard sense: the firm which posts the lower price receives the entirety of market demand (and ties are handled in whichever manner is most conducive to our analysis). Market demand is $Q_D = C - P$, where P is the prevailing price — the lower of the prices of the two firms — and CB is some constant reflecting the "strength" of demand (thought question: how could we introduce a parameter which would reflect the elasticity of demand?). "Strength" here is in quotes since it's not a solid economic definition, just an intuitive appeal to the fact that the larger C is, the greater the quantity demanded for a particular price.

There are two firms, A and B, facing the following cost curves:

$$C_A(q_A) = \begin{cases} q_A + 1 & \text{if } q_A > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$C_B(q_B) = 2q_B$$

That is, firm A faces a marginal cost of 1 and firm B faces a marginal cost of 2; however, firm A must also take into account a fixed cost of 1 for production.

We analyze possible prices as a sequence of cases.

- $P < 2$

  At first blush, it may seem obvious that $P < 2$ is not possible: since firm B will stay out when $P < 2$, firm A might have incentive to raise prices up to firm B's average cost function, $AC = 2$. However, we cannot forget that firm A has monopoly power in this situation; recall that monopolies sometimes have incentive to charge lower prices to stimulate demand.

  As a monopolist, firm A would face the following optimization problem:

$$\max_P P(C-P) - ((C-P)+1) = PC - P^2 - C + P - 1$$
$$\implies \quad 0 = C - 2P + 1$$
$$\iff \quad P = \frac{C+1}{2}$$
$$\implies \quad Q = \frac{C-1}{2}$$

  Notice that when $C < 1$, firm A's optimal quantity is negative. So when $C < 1$ neither firm will enter the market.

  What are firm A's profits? We compute

$$\pi_A = \left(\frac{C+1}{2}\right)\left(\frac{C-1}{2}\right) - \left(\frac{C-1}{2} + 1\right)$$
$$= \left(\frac{C-1}{2}\right)^2 - 1$$

  We see that profits will be negative when

$$\left(\frac{C-1}{2}\right)^2 < 1$$
$$\iff \quad (C-1)^2 < 4$$

Which will hold when $C < 3$ (assuming that $C > 0$ so that demand is necessarily positive).

So when $C < 3$, firm A's profits as a monopolist are necessarily negative. If $C > 3$, firm A's optimal price is $P = \frac{C+1}{2} > 2$ which allows firm B to enter and make positive profits. With this in mind, we will *never* see $P < 2$ in Nash equilibrium.

- $P = 2$

  When $P = 2$, it is evident that regardless of production level, $\pi_B = 0$. What are firm A's profits? By the market-demand equation we know $Q_D = C - 2$. So if firm A enters the market it will produce $C - 2$. Its profits are then

$$\begin{aligned}\pi_A &= 2(C-2) - ((C-2)+1) \\ &= (C-2) - 1 \\ \pi_A &= C - 3\end{aligned}$$

  It is evident that when $C < 3$, firm A makes $\pi_A < 0$ in profits from entering; in this case, then, it will prefer to stay out of the market. If $C > 3$, firm A makes $\pi_A > 0$ in profits from entering, and it will enter and produce $q_A = C - 2$.

  Then when $P = 2$, production is

$$(q_A, q_B) = \begin{cases} (C-2, 0) & \text{if } C > 3 \\ (0, C-2) & \text{otherwise} \end{cases}$$

  Now, we need to be careful with the above statement. If $P = 2$ and firm B is serving the entire market, it must be that $C < 3$. Then firm A does not want to enter, and firm B may be able to earn positive profits by raising the market price. This is the case we analyse below.

- $P > 2$

  If $P > 2$, firm B is serving the entire market. To see this, suppose that $P > 2$ and firm A serves the market. Then if firm B posts a price P' such that $2 < P' < P$ it will obtain the entire market and earn positive profits. Then firm A cannot be serving the market at $P > 2$ in equilibrium.

  If firm B enters the market as a monopolist, it solves the following optimization:

$$\begin{aligned} & \max_P P(C-P) - 2(C-P) = PC - P^2 - 2C + 2P \\ \implies & \quad 0 = C - 2P + 2 \\ \iff & \quad P = \frac{C+2}{2} \\ \implies & \quad Q = \frac{C-2}{2} \end{aligned}$$

$$\begin{aligned}\pi_B &= PQ - C_B(Q) \\ &= \left(\frac{C+2}{2}\right)\left(\frac{C-2}{2}\right) - 2\left(\frac{C-2}{2}\right) \\ \pi_B &= \left(\frac{C-2}{2}\right)^2\end{aligned}$$

  Clearly, when $C < 2$ firm B's optimal quantity is $Q = \frac{C-2}{2} < 0$. Since this is outside what we consider reasonable production, firm B's optimal quantity when $C < 2$ is 0 (that is, it does not enter the market). When $C \geq 2$, firm B produces $Q = \frac{C-2}{2}$ and earns $(\frac{C-2}{2})^2$ in profits.

How can we piece these two analyses together? Suppose that $C > 3$. Then if firm B enters and sets its own price, it will post $P = \frac{5}{2} > 2$. Since firm A can post a price of 2 and earn positive profits, the market price will be 2 in this case.

Now suppose that $C < 2$. Then neither firm A nor firm B can obtain positive profits from entry, and both will opt to stay out of the market; since "staying out of the market" is an artifact of quantity competition, we note that these firms can effectively stay out of the market by mutually posting a price of $P = C$, thereby driving demand to 0.

What happens when $2 < C < 3$? We know that firm A's *optimal* profits from entry when $C < 3$ are negative, so firm A will certainly not enter. Since firm A stays out, firm B gets to price as a monopolist. Then as seen above, firm B produces $\frac{C-2}{2}$ and captures the entirety of the market. We tabulate these results below, in Figure 8.

	$C < 2$	$2 < C < 3$	$3 < C$
q_A	0	0	$C - 2$
q_B	0	$\frac{C-2}{2}$	0
P	C	$\frac{C+2}{2}$	2
π_A	0	0	$C - 3$
π_B	0	$(\frac{C-2}{2})^2$	0

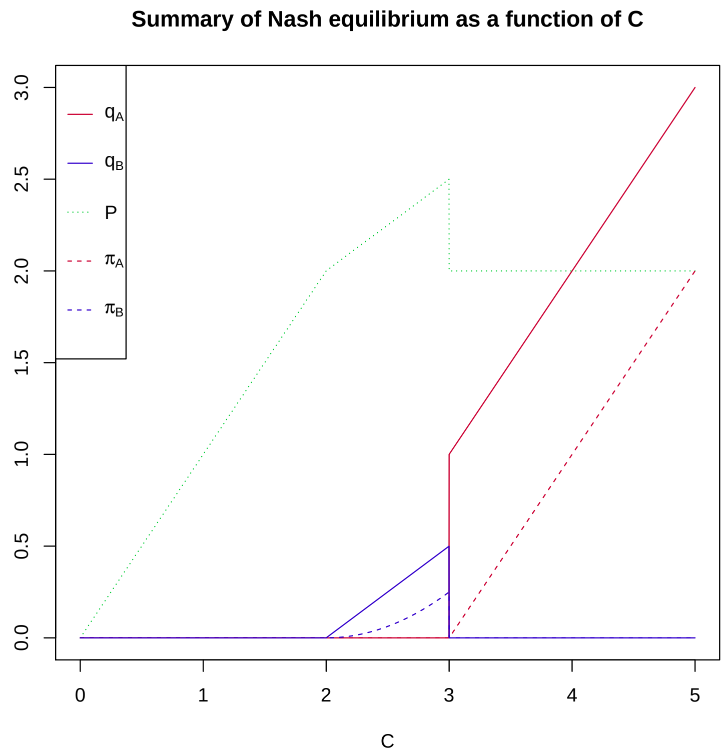

Figure 8: production quantities, prices, and profits for various "strengths" of demand C

Now, this model has some nice coincidences which make analysis more simple than it maybe should be. So let's change a parameter to see what changes.

Version 2

Retain the model setup from the previous question, except reduce firm A's fixed cost so that

$$C_A(q_A) = \begin{cases} q_A + \frac{1}{4} & \text{if } q_A > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let's consider our previous cases:

- $P < 2$

  Notice that firm B will not produce when $P < 2$. So if any entry occurs, it must be firm A entering as a monopoly. It will then solve

$$\begin{aligned} & \max_P P(C-P) - \left((C-P) + \frac{1}{4}\right) = \max_P PC - P^2 - C + P - \frac{1}{4} \\ \Longrightarrow \quad & 0 = C - 2P + 1 \\ \Longleftrightarrow \quad & P = \frac{C+1}{2} \\ \Longrightarrow \quad & Q = \frac{C-1}{2} \end{aligned}$$

  Then so long as $C > 1$, the optimal quantity is positive. We then continue and check whether or not the firm can earn positive profits at this level of production.

$$\begin{aligned} \pi_A &= \left(\frac{C+1}{2}\right)\left(\frac{C-1}{2}\right) - \left(\frac{C-1}{2} + \frac{1}{4}\right) \\ &= \left(\frac{C-1}{2}\right)^2 - \frac{1}{4} \end{aligned}$$

  Profits will then be positive so long as

$$\begin{aligned} & \left(\frac{C-1}{2}\right)^2 > \frac{1}{4} \\ \Longleftrightarrow \quad & (C-1)^2 > 1 \end{aligned}$$

  So firm A has a profitable entry strategy so long as $C > 1$. At this level, it will choose $P = \frac{C+1}{2}$.

- $P = 2$

  Again, if this is the case then $\pi_B = 0$ necessarily. Firm B is indifferent between producing and not. If firm A produces, it will receive

$$\begin{aligned} \pi_A &= 2(C-2) - \left((C-2) + \frac{1}{4}\right) \\ &= C - \frac{9}{4} \end{aligned}$$

  Then when $C > \frac{9}{4}$, firm A can obtain positive profits by meeting firm B's marginal cost.

- $P > 2$

  When $P > 2$, it must be that firm A is staying out of the market. Then firm B is operating as a monopolist. If this occurs, firm B solves the same problem as in Version 1 of this model; the firm's strategy results in

$$\begin{aligned} & P = \frac{C+2}{2} \\ \Longrightarrow \quad & Q = \frac{C-2}{2} \\ \Longrightarrow \quad & \pi_B = \left(\frac{C-2}{2}\right)^2 \end{aligned}$$

  Again, this strategy in monopoly is valid so long as $C > 2$.

What are the implications for this setup on equilibrium? Notice again that when $C > 3$, firm A can obtain a positive profit by undercutting firm B and setting $P = 2$. For all $C < 3$, firm A's optimal strategy in monopoly is to set $P = \frac{C+1}{2}$; since $C < 3$, it is evident that this $P < 2$. So at the optimum, firm A still undercuts firm B! Since firm A's monopolistic production keeps B from entering, this must be the Nash equilbrium outcome.

In this specification of the model, firm B only affects firm A's strategy when $C < 3$, driving firm A's monopoly pricing down to firm B's average cost function. Tabulating results, we find

	$C < 1$	$1 < C < 3$	$3 < C$
q_A	0	$\frac{C-1}{2}$	$C-2$
q_B	0	0	0
P	C	$\frac{C+1}{2}$	2
π_A	0	$(\frac{C-1}{2})^2 - \frac{1}{4}$	$C - \frac{9}{4}$
π_B	0	0	0

Figure 9: production quantities, prices, and profits for various "strengths" of demand C

Version 3

Retain the same model setup as in Version 1 (not Version 2!), except we increase the marginal cost of both firms so that

$$C_A(q_A) = \begin{cases} 1 + 2q_A & \text{if } q_A > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$C_B(q_B) = 4q_B$$

Essentially, we are now disadvantaging firm B further against firm A for larger production values.

Since firm B's marginal cost is now higher, the natural settling level for prices under large production quantities is higher. That is, when demand is sufficiently "strong" (C is sufficiently large) to keep firm B from undercutting, firm A will need to price at firm B's average cost, $P = 4$. We then look at the following cases:

- $P < 4$

  When $P < 4$, it must be that firm A captures all of the market; since the market price is below firm B's average cost, there is no way that firm B can make anything other than a loss in this situation. Then firm A solves the monopolist's problem,

$$\max_P P(C-P) - (2(C-P)+1) = \max_P PC - P^2 - 2C + 2P - 1$$
$$\implies \quad 0 = C - 2P + 2$$
$$\iff \quad P = \frac{C+2}{2}$$
$$\implies \quad Q = \frac{C-2}{2}$$

  So this production plan is valid so long as $C > 2$.

Profits in this situation are

$$\begin{aligned}\pi_A &= \left(\frac{C+2}{2}\right)\left(\frac{C-2}{2}\right) - \left(2\left(\frac{C-2}{2}\right)+1\right) \\ &= \left(\frac{C-2}{2}\right)^2 - 1\end{aligned}$$

Then maximized profits will be positive so long as

$$\begin{aligned} & \left(\frac{C-2}{2}\right)^2 > 1 \\ \iff \quad & (C-2)^2 > 4\end{aligned}$$

So profits from monopolistic production are positive when $C > 4$.

- $P = 4$

  As before, when $P = 4$ we must have $\pi_B = 0$ regardless of whether or not firm B serves the market. If firm A enters to capture market demand its profits will be

  $$\begin{aligned}\pi_A &= 4(C-4) - (2(C-4)+1) \\ &= 2C - 9\end{aligned}$$

  So it will earn positive profits so long as $C > \frac{9}{2}$.

- $P > 4$

  When $P > 4$, it must be that firm A is kept out of the market; otherwise, it would undercut firm B to firm B's average cost of 4. Firm B's monopoly problem is now

  $$\begin{aligned} & \max_P P(C-P) - 4(C-P) = \max_P PC - P^2 - 4C + 4P \\ \implies \quad & 0 = C - 2P + 4 \\ \iff \quad & P = \frac{C+4}{2} \\ \implies \quad & Q = \frac{C-4}{2}\end{aligned}$$

  This production plan is valid so long as $C > 4$. Profits under this production plan are

  $$\begin{aligned}\pi_B &= \left(\frac{C+4}{2}\right)\left(\frac{C-4}{2}\right) - 4\left(\frac{C-4}{2}\right) \\ &= \left(\frac{C-4}{2}\right)^2\end{aligned}$$

What strategy defines equilibrium? We have seen that when $C < 4$, neither firm can earn a positive profit. Further, when $4 < C < 6$, firm A would like to set a monopoly price of $\frac{C+2}{2}$ while firm B would like to set a monopoly price of $\frac{C+4}{2}$. Notice that for C in this range, $\frac{C+2}{2} < 4$ so that firm A, by pricing at its monopoly level naturally undercuts firm B. Lastly, for $C > 6$ firm A will be forced to set price $P = 4$ to undercut firm B.

This equilibrium is tabulated as

	$C < 4$	$4 < C < 6$	$6 < C$
q_A	0	$\frac{C-2}{2}$	$C - 4$
q_B	0	0	0
P	C	$\frac{C+2}{2}$	4
π_A	0	$(\frac{C-2}{2})^2 - 1$	$C - \frac{9}{2}$
π_B	0	0	0

Figure 10: production quantities, prices, and profits for various "strengths" of demand C

Version 4

As a last alteration, consider Version 1 but *increase* firm A's fixed cost so that

$$C_A(q_A) = \begin{cases} q_A + 4 & \text{if } q_A > 0 \\ 0 & \text{otherwise} \end{cases}$$

We return to our standard categorization of possible market prices.

- $P < 2$

  Firm B is unwilling to produce if $P < 2$, so if firm A chooses to produce it prices as a monopoly. Its problem is then

  $$\begin{aligned} & \max_P P(C-P) - ((C-P)+2) = \max_P PC - P^2 - C + P - 4 \\ \implies \quad & 0 = C - 2P + 1 \\ \iff \quad & P = \frac{C+1}{2} \\ \implies \quad & Q = \frac{C-1}{2} \end{aligned}$$

  As before, this production plan is valid so long as $C > 1$.

  Profits under this production plan are

  $$\begin{aligned} \pi_A &= \left(\frac{C+1}{2}\right)\left(\frac{C-1}{2}\right) - \left(\frac{C-1}{2} + 4\right) \\ &= \left(\frac{C-1}{2}\right)^2 - 4 \end{aligned}$$

  Profits will be positive when

  $$\begin{aligned} & \left(\frac{C-1}{2}\right)^2 > 4 \\ \iff \quad & (C-1)^2 > 16 \end{aligned}$$

  So profits are positive when $C > 5$.

- $P = 2$

  As before, firm B necessarily earns a profit of $\pi_B = 0$. If firm A serves the market, its profits are

  $$\begin{aligned} \pi_A &= 2(C-2) - ((C-2)+4) \\ &= C - 6 \end{aligned}$$

  Then so long as $C > 6$, firm A can earn positive profits by setting $P = 2$.

- $P > 2$

  The case of $P > 2$ is again the case of firm B pricing as a monopoly. Its problem is as in Version 1, resulting in a strategy defined by

$$\begin{aligned} & P = \frac{C+2}{2} \\ \implies \quad & Q = \frac{C-2}{2} \\ \implies \quad & \pi_B = \left(\frac{C-2}{2}\right)^2 \end{aligned}$$

  Again, this strategy in monopoly is valid so long as $C > 2$.

How can we build this into a description of Nash equilibrium? Clearly, if $C < 2$ neither firm would like to enter the market (and we indicate this in a price-competitive framework by firms posting a price which zeros out demand; here, $P = C$). When $2 < C < 5$ firm A finds it impossible to earn positive profits no matter what, so firm B is free to price as it sees fit and obtains the whole of the market. When $C > 6$, firm A can earn positive profits by undercutting firm B to its average cost of $P = 2$ and will do so.

We are left then with the case of $5 < C < 6$. In this range, it is possible for both firm A and firm B to make profits if operating alone, as a monopoly. However, we will see that these firms need to underprice their optima in order to keep their rival out of the market.

For such a C, firm B can earn a profit so long as $P > 2$ (that is, so long as $Q < C - 2$); firm A's profit condition is slightly more complex. Firm A can earn a profit so long as

$$\begin{aligned} & P(C-P) - ((C-P)+4) > 0 \\ \iff \quad & PC - P^2 - C + P - 4 > 0 \\ \iff \quad & P^2 - (C+1)P + (C+4) < 0 \end{aligned}$$

Noting that this parabola is upward-facing, we apply the quadratic equation to obtain bounds on when this inequality will obtain.

$$P = \frac{C + 1 \pm \sqrt{(C+1)^2 - 4C - 16}}{2} = \frac{C + 1 \pm \sqrt{C^2 - 2C - 15}}{2}$$

When P lies between the lower and upper solutions to the zeros of the quadratic, firm A can earn a positive profit. Since $5 < C < 6$, it is evident that we care only about the lower bound of this solution. Then firm A can earn a profit by outpricing firm B when

$$\frac{C + 1 - \sqrt{C^2 - 2C - 15}}{2} < 2$$

Working through the algebra, we find that this will hold when

$$\begin{aligned} & C + 1 - \sqrt{C^2 - 2C - 15} < 4 \\ \iff \quad & C - 3 < \sqrt{C^2 - 2C - 15} \\ \iff \quad & C^2 - 6C + 9 < C^2 - 2C - 15 \\ \iff \quad & 24 < 4C \\ \iff \quad & 6 < C \end{aligned}$$

Then firm A can earn a profit from outpricing firm B when $C > 6$. Notice that this contradicts our assumption that $5 < C < 6$! Then for all C in this range, firm B can undercut firm A to its average cost.

Notice that firm A's average cost function is $\frac{Q+4}{Q}$. With market demand given by $Q_D = C - P$, firm A's average cost at the market-clearing quantity is

$$AC_A = \frac{C - P + 4}{C - P}$$

To undercut and keep firm A out of the market, firm B needs to set a price equal to firm A's average cost. This implies

$$\begin{aligned} & \frac{C - P + 4}{C - P} = P \\ \iff & C - P + 4 = CP - P^2 \\ \iff & P^2 - (C + 1)P + (C + 4) = 0 \end{aligned}$$

Unsurprisingly, this is identical to the quadratic from before, when we were interested in whether or not firm A could make a positive profit. Then the solution to firm B's pricing problem is identical to the condition at which firm A breaks even; from above, this is

$$P = \frac{C + 1 - \sqrt{C^2 - 2C - 15}}{2}$$

Piecing this all together, we obtain a description of Nash equilibrium as

	$C < 2$	$2 < C < 5$	$5 < C < 6$	$6 < C$
q_A	0	0	0	$C - 2$
q_B	0	$\frac{C-2}{2}$	$\frac{C-1+\sqrt{C^2-2C-15}}{2}$	0
P	C	$\frac{C+2}{2}$	$\frac{C+1-\sqrt{C^2-2C-15}}{2}$	2
π_A	0	0	0	$C - 6$
π_B	0	$\left(\frac{C-2}{2}\right)^2$	something ugly	0

Since this is gnarly, a graphical description is given in Figure 11.

Summary

The take-away from this series of related question is a process for solving price-competition problems. Consider first how firms react when they have monopoly power; in which situations can they earn positive profits under monopoly power? If such situations overlap — that is, if there are parameterizations of the model for which both firms have some way of obtaining positive profits — we need to consider how price competition will play out. Price competition will *always* reflect some nature of firms attempting to keep one another out of the market by cutting profits. Since profits are 0 when price equals average cost, in these situations we can expect to see one firm producing at the average cost curve of the other (and we assume that consumers properly choose the firm we'd like them to when both firms post the same price). As in Version 4 above, this process can become quite complicated, but the general algorithm should not change.

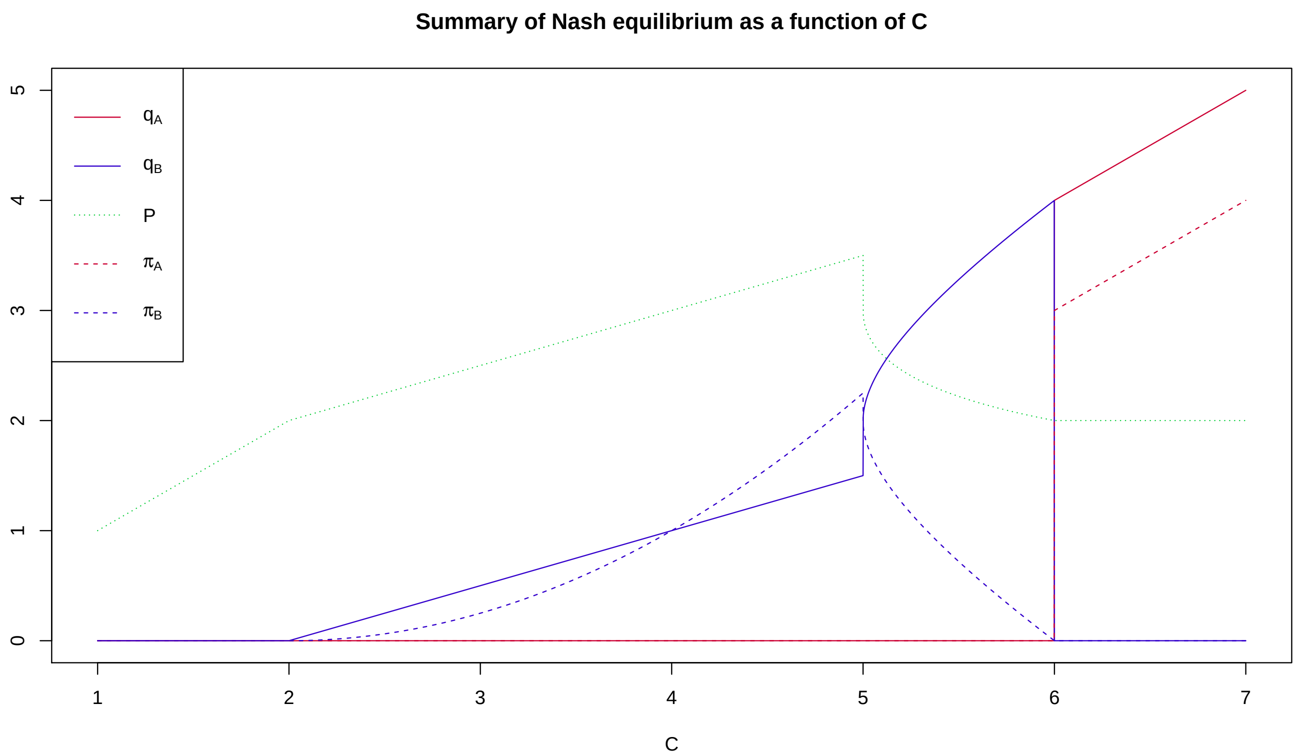

Figure 11: Nash equilibrium description for version 4 of the alternative pricing equilibrium game