Available online at www.sciencedirect.com

ScienceDirect

Journal of Economic Theory 185 (2020) 104954

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Uniform price auctions with a last accepted bid pricing rule

Justin Burkett [a], Kyle Woodward [b,*]

[a] *Georgia Institute of Technology, United States of America*
[b] *University of North Carolina at Chapel Hill, United States of America*

Received 7 December 2018; final version received 23 July 2019; accepted 17 October 2019
Available online 25 October 2019

Abstract

We model multi-unit auctions in which bidders' valuations are multidimensional private information. Under a natural constraint on aggregate demand we show that the *last accepted bid* uniform-pricing rule admits a unique equilibrium with a simple characterization: bids are identical to those submitted in a single-unit first price auction. The form of equilibrium bids suggests that last accepted bid uniform-pricing is a generalization of single-unit first-pricing: in both auctions winners pay the highest market clearing price. Contrasting the separating equilibrium of the last accepted bid auction, we show that equilibrium bids in pay as bid and first rejected bid uniform price auctions must pool information. Thus other common multi-unit auction formats cannot generalize single-unit first-pricing, in which equilibria do not pool information. The existence of a unique equilibrium implies that price selection may be an additional tool for avoiding the zero-revenue equilibria which exist in the first rejected bid uniform price auction. Finally, we show that equilibrium bids in our private information model are significantly flatter than in an analogous random supply model, suggesting that uniform price auction bids may not be as steep as commonly believed.

JEL classification: D44; D82

Keywords: Multi-unit auctions; Uniform pricing; First price auctions

* Corresponding author.
E-mail addresses: justin.burkett@econ.gatech.edu (J. Burkett), kyle.woodward@unc.edu (K. Woodward).

https://doi.org/10.1016/j.jet.2019.104954
0022-0531/

1. Introduction

In a multi-unit uniform price auction, bidders submit demand functions to a seller who awards m homogeneous units to the highest m bids at a single clearing price. This per-unit price may be the last accepted bid, the first rejected bid, or any intermediate amount.[1] These rules govern well-known large scale auctions, such as those run by the U.S. Treasury and the independent system operators in charge of electricity distribution. They are also used to model decentralized markets under the guise of competition in supply functions (Klemperer and Meyer, 1989)

Among common multi-unit auction formats, including pay-as-bid and Vickrey pricing, the uniform price rule is especially attractive because it awards homogeneous units at a homogeneous price.[2] The uniform price auction is fair, in the sense that bidders never pay less than other bidders for the same number of units won. If a goal of the auction is price discovery, the uniform price rule is a natural choice. This pricing rule also aligns well with textbook descriptions of decentralized markets (see, e.g., Kyle (1989); Vives (2011)).

However, a number of studies emphasize unappealing properties of the equilibria that may arise in these auctions. First, when bidders have private information, their bidding problem can be complex. A bid for any unit — except possibly for the first unit — may determine the clearing price, and hence influence the total amount paid. Consequently, bidders strategically reduce their bids below their demand, complicating inferences about the distribution of opposing bids when bidders have private information (Vickrey, 1961; Ausubel et al., 2014). Second, there may be equilibria that generate little to no revenue for the seller, and reveal no private information (Back and Zender, 1993). In these low-revenue equilibria bidders tacitly agree on a division of the goods and submit demand curves that implement this division at the lowest possible price. A third, closely-related issue is that there may be multiple equilibria, each with different allocations. Finally, and certainly true of many of the low-revenue equilibria, it has been argued that the equilibria of these auctions are generally inefficient (Ausubel et al., 2014).

In this paper, we show that incentives in uniform price auctions — and the undesirable properties they imply — are closely tied to the common modeling assumption that the per-unit price is the equal to the first rejected bid, and equilibrium predictions are dramatically affected if a different clearing price is selected. In practice, market clearing prices are computed as the price which would clear a Walrasian market, taking reported bid curves to be actual demand curves. With discrete goods this typically leaves some discretion to the seller: when m units are available, markets will (weakly) clear as long as the price is between the mth and $m+1$th highest bids. Existing literature on multi-unit auctions has tended to focus on the selection of the lowest market clearing price, which we term the *first rejected bid* (FRB).[3] We focus instead on the *last accepted bid* (LAB) pricing rule, in which the highest market clearing price is selected.[4]

[1] These rules are easily modified to accommodate bidders submitting supply curves to sell goods, sale of a divisible good and/or a seller using a non-constant supply (or demand) schedule.

[2] In a pay-as-bid auction, the highest m bids are awarded goods each winning bidder must pay the sum of their winning bids. By Vickrey pricing, we mean charging a bidder who wins k units the sum of the k highest rejected bids.

[3] The literature on divisible-good auctions has analyzed the last accepted bid auction and has produced results that contrast with ours. Notably, revenue can be arbitrarily low in a last accepted bid auction for a divisible good (Back and Zender, 1993). While there are other distinctions between the setting studied by Back and Zender (1993) and ours, perfect divisibility of the underlying good is crucial to the existence of low-revenue equilibria (see Remark 5).

[4] In a procurement auction these roles are reversed: the first rejected bid is the highest market clearing price and the last accepted bid is the lowest market clearing price. Our explicit focus in this paper is on sales auctions, but we use the

This apparently small change in the rules of the auction alters the alignment of bidding incentives and has dramatic effects on equilibrium behavior. In a first rejected bid auction the price is set by the bid for a supra-marginal unit, while in a last accepted bid auction the price is set by a bid for an exactly marginal unit. Then in a last accepted bid auction the probability a small increase in bid affects the resulting allocation is identical to the probability it affects the market clearing price, while in a first rejected bid auction these probabilities differ. While theoretical literature on uniform price auctions has noted the possibility of multiple pricing rules (see, e.g., McAdams (2003)), a natural reading suggests that the practical difference in price selection rules has been ignored due the large quantities typically sold in real-world auctions: when the number of units is large, there is typically not much difference between the level of the last accepted bid and the level of the first rejected bid.[5] We show that the implicit assumption that the exact location of the market clearing price is unimportant elides the different incentives induced by selection of one price or another. In spite of the fact that per-unit incentives do not change much, the aggregation of small changes yields a clear distinction between the two pricing mechanisms.

Our model of multi-unit auctions employs a novel private value framework, in which bidders have multidimensional private information about their demand for any number of units of an indivisible good.[6] Such values can arise if, for example, small agents with independent private values route orders through large institutional bidders. Our multidimensional private value model allows the flexible specification of each bidder's expected number of units demanded at each price, referred to as a bidder's mean demand curve, while imposing restrictions on the distribution of realized demand curves about this mean demand curve. We show that this model, applied to the last accepted bid uniform-pricing rule, is tractable and yields equilibrium behavior with several desirable properties, especially when compared to existing models.

We first show that, with two bidders, the equilibrium bids for *each* marginal unit take the form of bids in an asymmetric first price auction for a *single* unit. An immediate implication is that many of the results from the first price auction literature translate directly to our environment. For example, we are able to draw a connection between the relationship between two bidders' mean demand curves and how aggressively they bid in the auction. Extending the work of Maskin and Riley (2000), we classify bidders' mean demand curves as "strong" or "weak" and show that distributional weakness leads to aggressive bidding. If the two bidders' demands are symmetric, the bid curves can be solved analytically as in the first price auction. This analytical result extends cleanly to the case of n symmetric bidders, provided an algebraic condition on demand (market balance) is satisfied. Whereas the modern literature on multi-unit auctions emphasizes differences between equilibrium bidding strategies in multi-unit auctions and single-unit auctions, we find a close connection between our model of the uniform price auction and the standard model of the first price auction: the two auctions may have essentially identical equilibrium strategies.[7]

notions of last accepted and first rejected bids to capture the generality of the arguments without regard to the different meaning of "high" and "low" bids in sales and procurement contexts.

[5] These two points intersect in the analysis of divisible-good models: when submitted bids are continuous demand curves, there is no distinction between the highest and lowest prices clearing the market.

[6] Previous work, including McAdams (2007b) and Reny (2011), provides equilibrium existence and some worked examples of multi-unit uniform price auctions with multidimensional types. To the best of our knowledge ours is the first analytical characterization with demands exceeding two units.

[7] Historically, the literature on multi-unit auctions has sometimes assumed (without analysis) they were strategically analogous to single-unit auctions; see, e.g., Friedman (1991). Our results should not be mistaken for a return to this view.

This is in stark comparison to the first rejected bid and pay as bid auctions, which we prove cannot satisfy this equivalence.

Our closed-form results rely on an algebraic assumption on demand which we term *market balance*. In a balanced market with n bidders, each collection of $n-1$ bidders exactly demands the m units available for auction. Alternatively, any bidder's opponents exactly cover market supply. Under market balance the tie between last accepted bid and first price auctions is intuitive. A bidder's bid for unit k is relevant when it sets the market clearing price. This occurs when the bidder has outbid exactly k of her opponents' bids. When participants' values are given by ordered draws from a common distribution, this is equivalent to the bid for unit k defeating k draws from a common distribution, as long as the mapping from values to bids is unit-independent. Conditional on beating exactly these k draws the bidder pays her bid for this unit k times over, generating exactly the same incentives as the first price auction. As we discuss in Section 3, this "k-vs-k" intuition neatly captures the inability of first rejected bid and pay as bid auctions to generalize first price auctions.

The extent to which a last accepted bid auction generalizes a first price auction depends on the features which should be replicated. In the order statistic value model with private values, market balance is sufficient to imply an equilibrium in the last accepted bid auction which is essentially identical to equilibrium in an equivalent first price auction. If market balance is not satisfied, we find that qualitative features of equilibrium, such as separation and uniqueness, are satisfied by both last accepted bid and first price auctions (see Appendix D). Whether or not market balance is satisfied, these features are not shared by the first rejected bid or pay as bid formats. Thus while our closed-form results depend on the relationship between demand and supply, the shared equilibrium properties of the last accepted bid and first price auctions suggest that the pricing rule itself is the source of the comparison.

We emphasize the close connection between single-unit first price auctions and multi-unit last accepted bid auctions by comparing the last accepted bid auction to the first rejected bid uniform price and pay as bid rules. The comparison to first rejected bid is natural: an auctioneer implementing a uniform price auction has a choice of pricing rules, and our results suggest that the last accepted bid rule is preferable. The comparison to pay as bid is also of economic interest. We show that it is impossible for a multi-unit pay as bid auction to generate equilibrium bids which are identical to single-unit first price equilibrium bids, giving further evidence for our claim that last accepted bid pricing generalizes single-unit first pricing. Since multi-unit auctions are typically implemented as uniform price or pay as bid, our comparison of payment rules provides a guide for avoiding some undesirable properties of the first rejected bid and pay as bid formats by implementing a last accepted bid auction instead.[8]

In our comparison of auction formats, we first show that the first rejected bid and pay as bid auctions do not admit separating equilibria. Solving analytically for equilibrium in the first rejected bid and pay as bid auctions involves the determination of "pooled intervals" in the underlying type space, which imply regions over which the first order conditions cannot be naïvely applied. In the first rejected bid auction, pooling arises for relatively low valuations for which

As stated above, our results suggest a strategic connection between single-unit first price auctions and multi-unit last accepted bid auctions.

[8] The desirability of a particular multi-unit auction format depends on the desired features of equilibrium. We show that last accepted bid auctions can be tractable for participants, may be point identified, and that the pricing rule can eliminate low-revenue equilibria. To the extent that the auctioneer finds these properties desirable, the last accepted bid format may be useful.

the marginal gain associated with an increase in winning probability is outweighed by the increased probability of setting the market price. In the pay as bid auction, pooling arises due to the constraint that bids be weakly decreasing while agents would sometimes prefer to submit nonmonotone bid schedules. In both cases this provides a clear contrast with the unique separating equilibrium we find in the last accepted bid auction. All equilibria in the first rejected bid and pay as bid auctions are inefficient, in comparison with the unique and efficient separating equilibrium we characterize in last accepted bid auctions with market balance.[9]

Second, we show that the zero revenue, collusive-seeming equilibria of the first rejected bid auction cannot be sustained in the last accepted bid auction. In a first rejected bid auction, participants can tacitly agree on an allocation and bid aggressively for quantities they do receive, and nonaggressively for quantities they do not receive. Because the market price is set by the bid for a supra-marginal quantity, any deviation to obtain a greater quantity requires a dramatic increase in payment, which is not optimal. In the last accepted bid auction the market price is set by the bid for an exactly marginal quantity. Then if a bid profile guarantees a low clearing price there are strong incentives to increase bids, and there does not exist a generically low-revenue equilibrium. A subtle change in the uniform pricing rule can have beneficial effects on equilibrium selection, in line with other results on removing these equilibria with small adjustments to the uniform price model; we discuss related literature in Section 1.2.

Finally, we examine the role of private information in our model by analyzing the large-supply limit our unique equilibrium. This allows a clean comparison of uniform price auctions with multidimensional private information to much of the literature on uniform price auctions, which assumes common information but random supply (see Klemperer and Meyer (1989); Ausubel et al. (2014), and others): when market supply is large, private information in our model essentially disappears.[10] Furthermore, since quantities are effectively divisible when aggregate supply is large, there is no difference between first rejected and last accepted bid pricing, holding fixed a profile of (continuous) demand curves.[11]

In the large-supply limit, bids in our private information model are shallower than in common information models with random supply. The slope of equilibrium bids relates to per-unit incentives, and the question of who bidders are competing against. In existing models with full information, bids for high-value units compete against bids for high-value units and bids for low-value units compete against bids for low-value units, while in our model bids for high-value units compete against bids for low-value units, and vice-versa. Bidders with high values bid more aggressively against other high-value bidders than against low-value bidders, and bids are steeper in the former case than in the latter. This adds an important caveat to the common understanding that bids in uniform price auctions will be relatively steep: existing results have rested on the manner in which randomness enters the auction.

Taken as a whole, our results suggest that care be paid to the selection of salient features of auction models. For example, bidders report truthfully in the canonical equilibrium of a single-

[9] The logic establishing nonseparation in pay as bid auctions is familiar from McAdams (2008), which shows that locally-constant bids cannot be inverted to unique values. Our results are complementary, since we show that all equilibria of pay as bid auctions have locally-constant bids for some type realizations. Our results on pooling in first rejected bid auctions are distinct, as they imply that allocation probabilities are discontinuous.

[10] With an infinite number of draws, the xth percentile draw is the inverse of the distribution of values at this percentile.

[11] Since the equilibria in divisible-good uniform price auctions studied in the literature are ex post equilibria, comparisons between our large-and-fixed supply limit and the divisible-and-random supply models in the literature are well-founded.

unit second price auction; it is known that the optimality of truthful reporting does not extend to multi-unit first-rejected bid pricing (see above, and also Back and Zender (1993), Engelbrecht-Wiggans and Kahn (1998), Wang and Zender (2002), and Ausubel et al. (2014) among many others). The optimality of truthful reporting in a single-unit second price auction derives from its equivalence to a single-unit Vickrey auction. With multiple units this equivalence disappears. If the defining characteristic of a single-unit second price auction is truthful reporting, the Vickrey format is the multi-unit generalization. In a similar way, it is also known that the intuitive behavior in a single-unit first price auction does not translate to multi-unit pay as bid auctions, in spite of bidders paying their bids in both settings (see, e.g., Woodward (2016)). Our results show strategic equivalence between single-unit first price auctions and multi-unit last accepted bid auctions, suggesting that the strategically salient feature of these auctions is the selection of the highest market clearing price. To our knowledge this has gone unaddressed in the literature.

1.1. Illustration: two bidders and two goods

A simple example of our model helps make the results concrete. There are $n = 2$ bidders, $i \in \{1, 2\}$, each with demand for $m_i = 2$ units. An auctioneer is selling $m = 2$ units in a multi-unit auction. He solicits a weakly decreasing demand curve, b^i, from each of the bidders, and awards the two units to the agent(s) submitting the two highest bids.

Bidders have independent private values, and bidder i's value for her kth unit is denoted v^i_k. For each bidder, v^i is determined by ordering two independent draws from a $\mathcal{U}(0, 1)$ distribution; in particular, v^i_k is marginally distributed according the kth order statistic of a uniform distribution on $[0, 1]$, $v^i_k \sim \mathcal{U}_{(k)}(0, 1)$. Given a bid function b^i_k mapping bidder i's values to a bid for the kth unit, denote the inverse bid function mapping bids to values by φ^i_k.[12]

Denote the marginal distribution of v^i_k by $F_{(k)}$. Because values are distributed as order statistics,

$$F_{(1)}(x) = x^2, \text{ and } F_{(2)}(x) = 2x - x^2.$$

Under a last accepted bid payment rule, bidders pay the second-highest bid submitted for each unit they receive. Under a first rejected bid payment rule, bidders pay the third-highest bid submitted for each unit they receive. We defer discussion of pay as bid auctions, in which bidders pay their submitted bid for each unit they receive, until later in the paper.

1.1.1. Last accepted bid

In LAB, three statistical events affect bidder i's interim utility. First, bidder i can win 2 units; this occurs when $b^i_2 \geq b^{-i}_1$, and the market-clearing price is b^i_2. Second, bidder i can win 1 unit while bidder $-i$ sets the price; this occurs when $b^i_1 \geq b^{-i}_1 > b^i_2$. Third, bidder i can win 1 unit and set the price; this occurs when $b^{-i}_1 > b^i_1 \geq b^{-i}_2$.

Written in terms of these events, interim utility in LAB is

$$u^i\left(b^i; v^i\right) = \left(v^i_1 + v^i_2 - 2b^i_2\right) \Pr\left(b^i_2 \geq b^{-i}_1\right)$$

[12] In the analysis of this example we elide some technical details and focus on strictly decreasing and differentiable strategies; in particular, bids are strictly increasing in value and are thus invertible, and tiebreaking is a probability-zero event, so there is no concern about allocations when bidders submit the same bid.

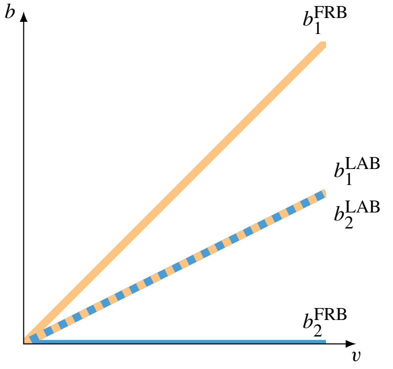

Fig. 1. Equilibrium bids in the example LAB and FRB auctions in Section 1.1.

$$+\left(v_1^i-\mathbb{E}\left[b_1^{-i}\middle|b_1^i\geq b_1^{-i}>b_2^i\right]\right)\Pr\left(b_1^i\geq b_1^{-i}>b_2^i\right)$$
$$+\left(v_1^i-b_1^i\right)\Pr\left(b_1^{-i}>b_1^i\geq b_2^{-i}\right).$$

We show in Appendix A that symmetric equilibrium (inverse) bid functions are

$$\varphi_1(b)=\varphi_2(b)=2b,\quad \text{and}\quad b_1(v)=b_2(v)=\frac{1}{2}v.$$

That is, equilibrium in LAB is exactly equilibrium in a standard first price auction for a single unit, when values are distributed uniformly on [0, 1]. These bids are depicted in the dashed lines in Fig. 1. It is immediate that this equilibrium is efficient.

1.1.2. First rejected bid

In FRB, three statistical events affect bidder i's interim utility. First, bidder i can win 2 units; this occurs when $b_2^i\geq b_1^{-i}$, and the market-clearing price is b_{-i}^1. Second, bidder i can win 1 unit and set the price; this occurs when $b_1^{-i}>b_2^i\geq b_2^{-i}$. Third, bidder i can win 1 unit while bidder $-i$ sets the price; this occurs when $b_1^i\geq b_2^{-i}\geq b_2^i$.

Written in terms of these events, interim utility in FRB is

$$u^i\left(b^i;v^i\right)=\left(v_1^i+v_2^i-2\mathbb{E}\left[b_1^{-i}\middle|b_2^i\geq b_1^{-i}\right]\right)\Pr\left(b_2^i\geq b_1^{-i}\right)$$
$$+\left(v_1^i-b_2^i\right)\Pr\left(b_1^{-i}>b_2^i\geq b_2^{-i}\right)$$
$$+\left(v_1^i-\mathbb{E}\left[b_2^{-i}\middle|b_1^i\geq b_2^{-i}>b_2^i\right]\right)\Pr\left(b_1^i\geq b_2^{-i}>b_2^i\right).$$

We show in Appendix A that in symmetric equilibrium bidder i's utility is weakly increasing in her bid for the first unit, provided the bid is below her value. Thus each bidder bids truthfully for her first unit. Subject to this result, each bidder's utility is weakly decreasing in her bid for the second unit. Then in the unique symmetric equilibrium,

$$b_1(v)=v,\ \text{and}\ b_2(v)=0.$$

These bids are depicted in the solid lines in Fig. 1.

Because bids in the LAB auction are independent of the unit for which they are submitted — that is, because $b_k^i(v)=v_k/2$ for both units — outcomes are efficient. This contrasts strongly with the results of the FRB auction. Because second-unit bids are always zero, inefficient outcomes will arise whenever one agent's value for her first unit is below the other agent's value for her second unit (in this example this occurs with probability $1/3$).

In the remainder of this paper we demonstrate that certain of the above properties generalize. When market balance is satisfied — that is, when aggregate supply is perfectly covered by any bidders' opponents — there is an equilibrium of the LAB auction in which bids are independent of the unit for which they are submitted, implying efficiency. Moreover, this is the unique separating equilibrium. Contrariwise, we demonstrate that there is always information confounding and some degree of pooling in the FRB auction, thus all equilibria of the FRB auction are inefficient. We also demonstrate that the FRB auction always admits an equilibrium with zero expected revenue, while the LAB auction admits no such equilibrium.

1.2. Related literature

This paper builds on results from multi- and single-unit auctions, as well as divisible-good auctions. Results in multi-unit auctions are hampered by the intractability of the multidimensional analysis required to maximize bidder utility. Results that address this intractability typically manage to reduce the dimension of the bid space; successful efforts on this front include Vickrey (1961), Engelbrecht-Wiggans and Kahn (1998), Engelbrecht-Wiggans and Kahn (2002), Lebrun and Tremblay (2003), Bresky (2013), and Ausubel et al. (2014). Faced with this difficulty in the multi-unit problem, a thread of literature has analyzed divisible-good models in which the quantity at auction is infinitely divisible. Continuity of equilibrium outcomes ensures that where predictions are continuous, large multi-unit auctions will be approximated by divisible-good models. Nonetheless success in this approximation has been essentially limited to cases with constant marginal values (e.g., Wilson (1979), Back and Zender (1993), and Wang and Zender (2002)) or no private information (e.g., Holmberg (2009), Anderson et al. (2013), and Pycia and Woodward (2019)).[13]

We emphasize the connection between the last accepted bid multi-unit and first price single-unit auctions, which allows us to leverage many results in first price auctions to quickly derive theoretical properties of our model. Early investigations of equilibrium uniqueness in first price include Plum (1992), Bajari (2001), and Maskin and Riley (2003), and these results have been extended by Lebrun (2006), Kirkegaard (2009), and Chawla and Hartline (2013), among others. In our uniqueness results, we closely follow the approach of Lebrun (2006). Structurally, our results on asymmetric bidders are in line with Lebrun (1999) and Maskin and Riley (2000). In contrast with uniqueness, which derives from analogy to first price auctions, equilibrium existence is directly guaranteed by known theoretical results (McAdams (2003), McAdams (2006), Reny (2011), Woodward (2017), and others).

Our results continue the thread of research investigating how to improve mechanism outcomes without substantially altering the allocation mechanism. In single-unit auctions reserve prices can be used not only to induce more aggressive bidding behavior (Myerson (1981), and many others) but also to select away seller-pessimal equilibria (Graham and Marshall (1987), Lizzeri and Persico (2000), Blume and Heidhues (2004), Blume et al. (2009), Chassang and Ortner (2015),

[13] A related body of work looks at uniform price auctions as approximations of classical market behavior, where a single price dictates the payments to and from all participants (e.g., Kyle (1989), Klemperer and Meyer (1989), and Vives (2011)). These models apply uniform pricing, but reject the constraint that allocations must be positive.

Burkett and Woodward (2018)). In multi-unit auctions it is known that flexible supply — a generalization of a reserve price — can improve the seller's revenue (LiCalzi and Pavan (2005), McAdams (2007a)). Kremer and Nyborg (2004a,b) show that bid discretization and tiebreaking rules can be useful for eliminating underpricing equilibria in uniform price auctions, holding the market price fixed. Our results show that price selection itself can be a valuable tool for eliminating these undesirable equilibria, leaving the fundamentals of the auction essentially intact.[14]

Although we do not attempt an empirical exercise, our results are potentially useful for practitioners. The theoretical ambiguity of outcome rankings between pay as bid and uniform price auctions provides a natural opening for empirical work. Existing studies provide an ambiguous lesson in mechanism selection: depending on context, pay as bid outperforms uniform price (Fevrier et al. (2002), Kang and Puller (2008), and Marszalec (2017)), uniform price outperforms pay as bid (Armantier and Sbaï (2006), Castellanos and Oviedo (2008), and Armantier and Sbaï (2009)), there is no statistical difference between the two (Hortaçsu and McAdams (2010)), and there is no practical difference between the two (Hortaçsu et al. (2018)). Notably, counterfactual estimates typically rely on bounding best response behavior, as in Hortaçsu and McAdams (2010). Our tractable model of bidder behavior may provide a method for making these predictions more precise. For example, our results on separating equilibria show that values in our model are point identified.

The remainder of the paper is organized as follows. Section 2 introduces the general model. In Section 3, we characterize equilibrium in the last accepted bid uniform price auction, and presents results on equilibrium uniqueness. Section 4 contrasts the informational and uniqueness properties of last accepted bid uniform price auctions with first rejected bid uniform price and pay as bid auctions, and provides a large market analysis. Section 5 concludes.

2. Model

An auctioneer is selling m units of a homogeneous good to n risk-neutral bidders who, with probability 1, have strictly positive aggregate demand for at least m units.[15] Bidder i values units according to the ordered realizations of m_i independent draws from the continuous distribution $F^i : [0, 1] \to [0, 1]$ with density f^i. The ordering ensures that marginal values are weakly declining for every realization. For example, the bidder's marginal value for the first unit is the first order statistic of m_i independent draws from F^i. When bidders are symmetric, $F^i = F$ for all i and some F. We denote the ordered vector of bidder i's valuations by v^i, so that v^i_k is her value for her kth unit. By definition, $v^i_1 \geq \cdots \geq v^i_{m_i}$. For our closed-form results we employ an assumption of *market balance*, requiring that $m = \sum_{j \neq i} m_j$ for any bidder i.[16] Market balance is further discussed in Section 3.

Our value model can also account for stochastic total demand. For simplicity we assume that the underlying distributions F^i are continuous and that densities f^i are defined everywhere,

[14] To extend the single-unit analogy, first price auctions typically have unique equilibria while second price auctions typically have many equilibria, some of which admit zero revenue. Therefore, in single-unit auctions price selection can be used as a tool for eliminating these zero-revenue equilibria.

[15] We later consider the (stochastic) possibility of insufficient demand. This does not meaningfully complicate or change our results.

[16] Market balance has a simpler algebraic formulation, $m = (n - 1)m_i$. We express this condition as a summation to emphasize that each bidder's opponents can exactly cover the market.

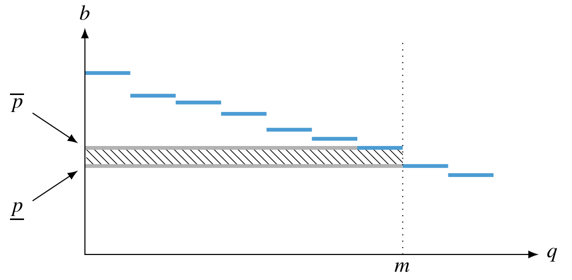

Fig. 2. Maximum and minimum market clearing prices displayed on an aggregate demand curve, $D(q) = \inf\{p : \#\{(i,k) : b_k^i \geq p\} < q\}$, when $m = 7$ units are available.

but all our results go through in the presence of a mass point at 0, the lowest possible value.[17] Since per-unit values are given by order statistics of m_i draws from this distribution, this implies that each agent's total demand — the number of units for which she has a strictly positive value — is given by a binomial distribution. Thus our model can fit natural settings in which demand is roughly known, subject to some small variance. With stochastic total demand market balance is satisfied if (deterministic) market balance is satisfied at the maximum realization of demand.

We consider sealed-bid auctions, where bidders submit weakly decreasing demand vectors (bid curves) to the auctioneer. Bidder i submits a weakly positive demand vector b^i, so that b_k^i is her bid for her kth unit. Where helpful, we will take a mechanism design approach and consider bids as functions of bidders' private values, $b_k^i \equiv b_k^i(v^i)$. Without a reserve price, the auctioneer allocates the available units to the m highest bids.[18,19,20] Denote the maximum and minimum market clearing prices by $\overline{p}$ and $\underline{p}$, respectively, where

$$\overline{p} = \min\left\{p : \#\left\{(i,k) : b_k^i \geq p\right\} \leq m\right\},$$
$$\underline{p} = \max\left\{p : \#\left\{(i,k) : b_k^i \geq p\right\} \geq m\right\}.$$

The distinction between these two prices is illustrated in Fig. 2.

Each bidder is risk-neutral and her utility is quasilinear in payments. Conditional on allocation q_i and payment t_i, bidder i's ex post utility is

[17] Indeed, some of our results are strengthened in the presence of such a mass point. Our theoretical results rely on results developed in the study of first price auctions. Equilibrium uniqueness in a first price auction, for example, is guaranteed when the distribution of values is log-supermodular, or when there is a mass point at the minimum possible value (Lebrun, 2006).

[18] Bid monotonicity is a constraint typically observed in practice. However, under the assumption that the auctioneer accepts bids in decreasing order, bid monotonicity is without loss of generality.

[19] Where the mth highest bid is not well defined some form of tiebreaking or rationing is necessary. Because the tiebreaking rule is not of importance to our analysis, we leave it unspecified. This point has been noted in the multi-unit and divisible-good auction literature; see, e.g., Häfner (2015).

[20] When a nontrivial reserve price is present, bids are accepted in decreasing order until either all m units are allocated or there are no remaining bids weakly above the reserve price. For the most part our results are not meaningfully affected by the presence or absence of a reserve price (in light of what is known of behavior in single-unit auctions with reserve prices), so we specify a model without a reserve price to avoid unnecessary technicalities.

$$u^i\left(q_i, t_i; v^i\right) = \left[\sum_{k=1}^{q_i} v_k^i\right] - t_i$$

We focus most of our attention on the *last accepted bid* (LAB) uniform-pricing rule, in which each bidder pays the same price for each unit she obtains, and this price is equal to the mth highest bid; this is equivalent to clearing the market at $\overline{p}$, the highest market clearing price, and implies a transfer of $t_i^{\text{LAB}}(q_i) = \overline{p}q_i$. For mechanism comparisons, we also discuss the *first rejected bid* (FRB) uniform-pricing rule, where the per-unit price is the $(m+1)$th highest bid (equivalent to the lowest market clearing price) and the transfer is $t_i^{\text{FRB}}(q_i) = \underline{p}q_i$, and the *pay as bid* (PAB) pricing rule, in which for each unit a bidder receives she pays her bid for this specific unit, and the transfer is $t_i^{\text{PAB}}(q_i) = \sum_{k=1}^{q_i} b_k^i$.

Market clearing implies that bidder i receives unit k if and only if her opponents receive (in aggregate) less than $m-k+1$ units.[21] It is helpful to consider bidder i competing for her kth unit against the aggregate demand of her opponents for $m-k+1$ units. Let H_{m-k+1}^{-i} be the marginal distribution of her opponents' $m-k+1$th highest bid, and let h_{m-k+1}^{-i} be the associated density (where well-defined).

2.1. Order statistics as demand curves

Our order statistics model has a natural interpretation in terms of bidders' mean demand curves. Fixing a uniform price p, the expected number of units demanded by bidder i is $(1 - F^i(p))m_i$.[22] The specification of the mean demand curve is flexible, since F^i is only constrained to be differentiable; however the distribution of demand curves about the mean is determined by the properties of the order statistic model.[23]

In Section 4.5 we investigate equilibrium as supply becomes large, modeled as goods becoming divisible. Specifically, each bidder demands up to a quantity of 1, which is divisible in $1/m$ increments (this is a horizontal rescaling of our model, dividing each bidder's utility by m). With an infinite number of value draws there is no distinction between mean demand and actual demand, and we refer to the inverse underlying value distribution $[F^i]^{-1}(1-x)$ as *asymptotic demand*. Given a maximum demand of 1, x corresponds to the fulfillment ratio, and asymptotic demand is the marginal value at this ratio.

3. Equilibrium properties

We begin by deriving first order conditions for best response behavior, under the assumption that equilibrium inverse bid functions are differentiable; we later verify this well-behavedness assumption. Using these first order conditions, we describe equilibrium bidding strategies in balanced markets when there are two bidders, and obtain a closed-form characterization when all bidders are symmetric. In the two-bidder case we show that equilibrium is unique in a natural class, and provide comparative statics on bid distributions when bidders are asymmetric.

---

21 With a reserve price the "only if" is still valid, but the "if" may fail. Nonetheless the competition faced for unit k is against opponents' aggregate demand for $m-k+1$ units.

22 For a fixed price, the number of units demanded out of a maximum of m_i is a random variable with a binomial distribution with probability of success (demanding a unit at price p) given by $1 - F^i(p)$.

23 For example, the observations in Footnote 22 imply that the variance of the number of units demanded at price p must be $F^i(p)(1-F^i(p))m_i$, or large for intermediate prices and small for prices near 0 or 1.

A bidder's interim utility is the probability-weighted sum of her utility for ex post allocations. Conditional on receiving exactly k units, two events determine the bidder's ex post utility: either her bid for unit k, b_k^i sets the market clearing price, or the market clearing price is below b_k^i and above b_{k+1}^i.[24] Let q_i denote the quantity bidder i receives. Taking as given her opponents' bid functions b^{-i}, bidder i's interim utility is given by

$$
\begin{aligned}
u_k^i\left(b; v^i\right) = \sum_{k=1}^{m_i}\Bigg(& \left(\sum_{\ell=1}^{k} v_k^i\right)\Pr\left(q_i = k\right) \\
& - k b_k^i \Pr\left(q_i = k, b_k^i \text{ sets price}\right) \\
& - k\mathbb{E}\left[b_{m-k+1}^{-i} \middle| q_i = k, b_{m-k+1}^{-i} \text{ sets price}\right]\Pr\left(q_i = k, b_{m-k+1}^{-i} \text{ sets price}\right)\Bigg).
\end{aligned}
\tag{1}
$$

The relevant events have simple expressions in terms of the distribution of opponents' bids[25]:

- Bidder i wins k units when her bid for her kth unit is larger than her opponents' bid for their $m-k+1$th aggregate unit, and her bid for her $k+1$th unit is less than her opponents' bid for their $m-k$th aggregate unit. Then

$$\Pr\left(q_i = k\right) = H_{m-k+1}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_{k+1}^i\right).$$

- Conditional on winning k units, the bid b_k^i sets the market clearing price whenever it is the mth highest bid. Then b_k^i sets the market clearing price when her opponents' bid for their $m-k$th aggregate unit exceeds her bid for her kth unit, $b_{m-k}^{-i} > b_k^i$. Then

$$\Pr\left(q_i = k, b_k^i \text{ sets price}\right) = H_{m-k+1}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_k^i\right).$$

- Conditional on winning k units, the opponents' bid for the $m-k$th aggregate unit sets the market clearing price whenever it is the mth highest bid. From bidder i's perspective, bid b_{m-k}^{-i} sets the market clearing price whenever b_{m-k}^{-i} is between b_k^i and b_{k+1}^i. Then

$$
\begin{aligned}
&\mathbb{E}\left[b_{m-k+1}^{-i} \middle| q_i = k, b_{m-k+1}^{-i} \text{ sets price}\right]\Pr\left(q_i = k, b_{m-k+1}^{-i} \text{ sets price}\right) \\
&= \int_{b_{k+1}^i}^{b_k^i} b\, dH_{m-k+1}^{-i}(b).
\end{aligned}
$$

[24] For simplicity of notation, we do not separately model the endpoint conditions of receiving 0 or m_i+1 units. We assume that the phantom bid for the 0th unit, b_0^i is sufficiently high, and the phantom bid for the m_i+1th unit, $b_{m_i+1}^i$, is sufficiently low, generating bid distribution functions so that the bidder receives between 0 and m_i units.

[25] We assume here that there are no mass points in the bid distribution, hence it is not mathematically relevant whether our probabilities are given in terms of strict or weak inequality. We show later that it is without loss of generality to ignore mass points in equilibrium bid functions.

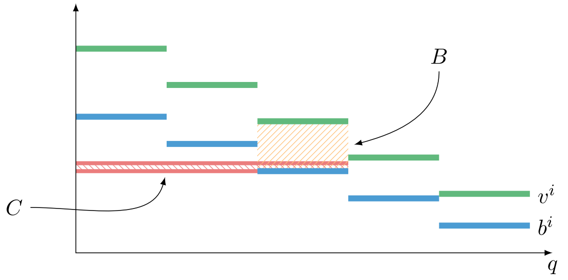

Fig. 3. Costs and benefits of a small increase in the bid for unit $k = 3$. Region B is the gains from a small increase in bid, the area between the value and bid for unit k; it is a discrete benefit, obtained with marginally greater probability. Region C is the costs from a small increase in bid, the area between the original and increased bids for unit k; it is a marginally greater cost, incurred with discrete probability. Note that the increase in payment is multiplied by the number of units which are obtained. Except for this multiplier, the tradeoffs are identical to a standard first price auction.

In terms of these expressions, interim utility (1) can be written as

$$u^i\left(b^i; v\right) = \sum_{k=1}^{m} v_k H^{-i}_{m-k+1}\left(b^i_k\right) - \left(H^{-i}_{m-k+1}\left(b^i_k\right) - H^{-i}_{m-k}\left(b^i_k\right)\right) k b^i_k$$
$$- k \int_0^{b^i_k} b\, dH^{-i}_{m-k} + (k-1) \int_0^{b^i_k} b\, dH^{-i}_{m-k+1}.$$

Ignoring for now the quantity monotonicity constraint — that bids must be decreasing in quantity — this allows for a surprisingly compact representation of the first order conditions for optimality.

Lemma 1 *(LAB first order conditions). In LAB, bidder i's first order condition for her bid for her kth unit is*

$$\underbrace{\left(v_k - b_k\right) dH^{-i}_{m-k+1}\left(b_k\right)}_{\textit{benefits}} = \underbrace{\left(H^{-i}_{m-k+1}\left(b_k\right) - H^{-i}_{m-k}\left(b_k\right)\right) k}_{\textit{costs}}. \tag{2}$$

Lemma 1 cleanly compares the costs and benefits of a marginal increase in bid. The benefits, on the left-hand side of equation (2), are exactly the margin on unit k times the marginal increase in probability of winning unit k. An increase in the bid for unit k cannot affect the probability of winning $k - 1$ units or $k + 1$ units. The costs, on the right-hand side of equation (2), are exactly the probability that bid b^i_k sets the market clearing price multiplied by k, to account for the fact that the small increase in bid is paid k times over, once for each unit won; this is illustrated in Fig. 3. If the quantity monotonicity constraint does not bind, this is the total effect of increasing the bid for unit k. On the other hand, if the quantity monotonicity constraint is binding a small increase in the bid for unit k may result in an increase in the bids for some units $k' < k$, and potentially allows increasing the bid for some units $k' > k$, and these effects would need to be accounted for. The monotonicity constraint is not binding in the equilibrium we construct, so equation (2) captures all first order effects.

In this section we constrain attention to strictly separating strategies, which we discuss in more depth in Section 4.

Definition 1 *(Strictly separating bids).* A bid function $\mathbf{b}^i$ is *strictly separating* if the inverse bid correspondence is at most single-valued; that is, for all type profiles v^i,

$$\#\left\{v : \mathbf{b}^i(v) = \mathbf{b}^i\left(v^i\right)\right\} = 1.$$

An equilibrium bid profile $(\mathbf{b}^i)_{i=1}^n$ is strictly separating if $\mathbf{b}^i$ is strictly separating, for all $i \in \{1, \ldots, n\}$.

In equilibria that are strictly separating bid curves preserve all information about the marginal values. In other words, bid curves are invertible given bid data — a useful property for empirical work. Because the monotonicity constraint on bid curves never binds, the optimization problem can be solved bid-by-bid. This reduces the complexity of the bidder's problem as well as the complexity of computationally solving for equilibrium. When strategies are not strictly separating, locally profitable deviations may not be feasible due to a binding monotonicity constraint, dramatically complicating equilibrium analysis.

We show below that utility is separable by unit. Then when the monotonicity constraint does not bind, bidding incentives are separable by unit. In multi-unit auctions with risk-neutral bidders, a bidder's utility for any allocation can be expressed as a sum of distinct per-unit utilities. Utility is modular (McAdams, 2003): a bidder's utility for allocation q_i is exactly her utility for allocation $q_i - 1$, plus her margin on unit q_i. This implies that bids for different units are co-determined only when the monotonicity constraint is binding.[26] Then when the monotonicity constraint is *not* binding, it is without loss of generality to analyze behavior in these auctions on a per-dimension basis, as a set of m independent optimization problems. Lemma 2 formalizes this separability in the auction formats we examine.

Lemma 2 *(Separability of multi-unit auctions). For each of the FRB, LAB, and PAB auctions there is a profile of utility functions $((u_k^i)_{k=1}^{m_i})_{i=1}^n$ such that ex post utility can be written as*

$$u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m_i} u_k^i\left(b_k^i, b^{-i}; v_k\right).$$

In each of these auctions, each dimensional utility function u_k^i satisfies increasing differences in (b_k, v_k). Conditional on opponent bids b^{-i}, interim utility can be written as

$$u^i\left(b^i; v\right) = \sum_{k=1}^{m_i} u_k^i\left(b^i; v_k\right).$$

Proof. See Appendix C. $\square$

Bids are strictly separating when the inverse bid function, mapping observed bids to underlying values, is well defined. It is immediate that in a strictly separating equilibrium, bids are strictly increasing in value. It is also true that in a strictly separating equilibrium the monotonicity constraint cannot bind. Suppose that the monotonicity constraint binds, so that bidder i's ideal (unconstrained) bid for unit k, b_k^i, is strictly less than her ideal (unconstrained) bid for unit $k+1$,

26 These models exhibit the standard IPV mechanism design monotonicity-in-value. Because this is a result and not a constraint, when we refer to binding monotonicity constraints we are referring to monotonicity in quantity.

$b^i_{k+1} > b^i_k$. Lemma 9 in the Appendix proves that the optimal constrained bid $\hat{b} \in (b^i_k, b^i_{k+1})$. Since utility is continuous in value, a small change in her underlying type — for example, a small decrease in v^i_k offset by a small increase in v^i_{k+1} — will leave the monotonicity constraint binding, and hold the optimal constrained bid constant. This implies noninvertibility of equilibrium bid functions, and thus in any strictly separating equilibrium the monotonicity constraint cannot (strictly) bind. This is discussed in more detail in Section 4.3.

A special case of separation arises when bidder i uses the same bid function for all units, $b^i_k \equiv b^i$. By construction the quantity monotonicity constraint will never bind, since values are weakly decreasing in quantity. Moreover, since the bidder's values are distributed as order statistics, her bids will also be distributed as order statistics. Let φ^i be bidder i's inverse bid function. Then

$$\Pr\left(b^i_k \leq b\right) = \Pr\left(b^i\left(v_k\right) \leq b\right) = \Pr\left(v^i_k \leq \varphi^i\left(b\right)\right) = F^i_{(k)}\left(\varphi^i\left(b\right)\right).$$

We now use equation (2) and presumed quantity-invariant bids to obtain explicit forms for equilibrium bids in two natural cases of market balance: two potentially asymmetric bidders, and any number of symmetric bidders. In each case market balance allows the generic opponent bid distribution functions H^{-i} to be conveniently factored. In Appendix D we analyze a model of an imbalanced market, and show that equilibrium is separating and unique but does not have a simple closed form. When market balance is not satisfied, equation (2) cannot be conveniently factored and a closed form for equilibrium bids is not possible. It can be shown that the bid for unit k cannot be independent of k, implying that bids are not distributed as order statistics. Equilibrium first order conditions in the market-imbalance case cannot be transformed to a single-dimensional differential equation, and are nondegenerately multidimensional. Given the existence of equilibrium and a necessary differential system, computational methods are necessary to obtain explicit equilibrium bids.

3.1. Two asymmetric bidders

Most of the literature on asymmetric first price auctions focuses on the two-bidder case. An analogous case in this model is set up as follows. Suppose that there are $n = 2$ bidders, $i \in \{1, 2\}$, who each value marginal units according to m draws each from F^i, where $F^1 \neq F^2$. Suppose that i's bids for each marginal unit are determined by the increasing, differentiable function $b^i(v^i_k)$ with inverse $\varphi^i(b)$, common across all units. Then bidder i wins unit k if and only if bidder $-i$ does not win unit $m - k + 1$. It follows that $H^{-i}_{m-k} = F^{-i}_{(m-k)} \circ \varphi^{-i}$, and equation (2) becomes

$$\begin{aligned}
&(v_k - b)\, f^{-i}_{(m-k+1)}\left(\varphi^{-i}\left(b\right)\right) d\varphi^{-i}\left(b\right) \\
&\quad = \left(F^{-i}_{(m-k+1)}\left(\varphi^{-i}\left(b\right)\right) - F^{-i}_{(m-k)}\left(\varphi^{-i}\left(b\right)\right)\right) k. \qquad (3)
\end{aligned}$$

Algebraic manipulation reduces (3) to the expression studied in Maskin and Riley (2000). If $b^1(v)$ and $b^2(v)$ are the equilibrium bid functions from the first price auction involving two bidders with corresponding value distributions F^1 and F^2, then setting $b^i_k = b^i(v^i_k)$ will satisfy bidder i's first order condition for good k when $\mathbf{b}^i(\mathbf{v}^i) = \left(b^i(v^i_k)\right)_{k\in\{1,\dots,m\}}$.

Proposition 1 *(Equilibrium in asymmetric LAB). With two bidders whose m marginal values are the order statistics from F^1 and F^2, let $b^1(v)$ and $b^2(v)$ be the equilibrium bid functions in the first price auction for a single unit with two bidders whose values are distributed according to*

F^1 and F^2. Then the strategies $\mathbf{b}^i(\mathbf{v}^i) = \left(b^i(v^i_k)\right)_{k\in\{1,..,m\}}$ constitute an equilibrium of the LAB auction.

Proof. Applying facts about order statistics, the costs in equation (2) (the right-hand side of (3)) are given by

$$\left(H^{-i}_{m-k+1}(b) - H^{-i}_{m-k}(b)\right)k = \binom{m}{m-k}\left(1 - F^{-i}\left(\varphi^{-i}(b)\right)\right)^{m-k} F^{-i}\left(\varphi^{-i}(b)\right)^{k} k.$$

The benefits in equation (2) (the left-hand side of (3)) are given by

$$\begin{aligned}&(v_k - b)\, dH^{-i}_{m-k+1}(b)\\ &= (v_k - b_k)\binom{m}{m-k}F^{-i}\left(\varphi^{-i}(b)\right)^{k-1}\left(1 - F^{-i}\left(\varphi^{-i}(b)\right)\right)^{m-k} kf^{-i}\\ &\quad\times\left(\varphi^{-i}(b)\right)d\varphi^{-i}(b).\end{aligned}$$

Then equation (2) becomes

$$(v_k - b_k)\, f^{-i}\left(\varphi^{-i}(b_k)\right)d\varphi^{-i}(b_k) = F^{-i}\left(\varphi^{-i}(b_k)\right).$$

It is clear that the candidate equilibrium bid, $b^i(v^i_k)$, satisfies the kth first order condition. Standard arguments establish that the partial derivative is negative (positive) for $b' > b_k$ ($b' < b_k$), which also means that the objective must be lower at the end points, $b^i(v^i_{k-1})$ and $b^i(v^i_{k+1})$, given by the monotonicity constraint. □

Equilibrium bid functions in Proposition 3 are exactly those in a two-bidder first price auction with private value distributions F^1, F^2, suggesting a connection between multi-unit LAB auctions and first price single-unit auctions. In both auctions the per-unit price is equal to the highest feasible market clearing price (according to strategically submitted bids). Crucially this price is equal to a bid which is actually awarded, hence when the bid for unit k is relevant to the bidder's payment it is paid exactly k times over; see the left panel of Fig. 4. When the bid for unit k is relevant to whether or not the bidder wins unit k it is because the bid exceeds k opposing bids and is exactly tied with one of these. Any of the k weakly lower opposing bids could be the highest, so, controlling for k bids being lower, there are k ways this tie can occur; see the right panel of Fig. 4. The k-fold increase in payment scales directly with the k ways in which a tie can occur, resulting in classical single-unit first price incentives.

The "k-vs-k" reasoning highlights the interplay between market balance, the order statistic model, and the last accepted bid payment rule. Alternate multi-unit auction formats, such as first rejected bid and pay as bid, cannot generate these same tradeoffs in this context. In a pay as bid auction, a winning bid for unit k has defeated k opponent bids, but is paid once over. This leads to more aggressive bidding for large quantities than for small quantities, and implies that equilibrium bids cannot take the form of those in a single-unit first price auction.[27] Incentives are also misaligned in first rejected bid auctions, where a bid which is paid k times over has defeated $k-1$ opponent bids: a bid sets the market clearing price only if it is the first losing bid. It is not possible for these terms to offset one another. Although market balance is essential to

[27] For a model parameterization in which a variant of the "k-vs-k" logic does apply to the pay as bid auction, see Engelbrecht-Wiggans and Kahn (2002).

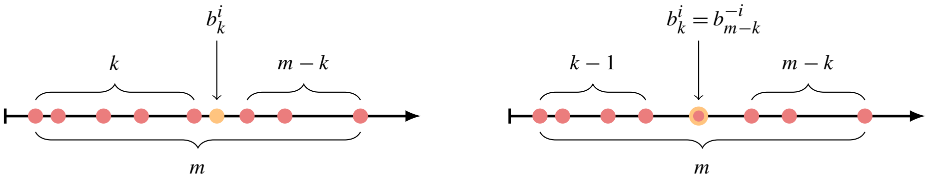

Fig. 4. Two situations in which the bid for unit k is relevant. In the left panel, the bid for unit k sets the market clearing price, and an increase in bid is paid k times (once for each unit received). In the right panel, the bid for unit k ties the opposing bid for unit $m-k$; conditional on (weakly) beating k opponent bids there are k ways this tie can occur. The k multipliers cancel, resulting in incentives familiar from single-unit auctions. This emphasizes why similar equilibria are not possible in the FRB auction, where the bid is paid $k-1$ times, or in the PAB auction, where the bid is paid once.

closed-form replication of first price bid curves, it is inessential to the argument that first rejected bid and pay as bid auctions cannot replicate important features of first price auctions. For further discussion, see Section 4.

3.1.1. *Bid aggressiveness*

Given the relation between equilibrium bidding in the LAB auction and bidding in asymmetric first price auctions, a number of results follow immediately. Instead of exhaustively listing them here, we emphasize their interpretation in this model. Recall that we may interpret the function $(1-F^i(p))m$ as bidder i's mean demand curve. It follows from the previous section, that $(1-F^i(\varphi^i(b)))m$ represents the mean number of bids placed by bidder i that exceed b in equilibrium, referred to as the mean quantity demanded in equilibrium. Bidder $-i$'s mean residual supply curve is therefore $F^i(\varphi^i(b))m$, which is proportional to the equilibrium bid distribution of a bidder with type distribution F^i in a first price auction.

The stochastic dominance properties used in the asymmetric first price auction literature have immediate analogues to properties of the mean demand curves in this model. For example, bidder i having weakly higher mean demand than bidder $-i$ at each price is equivalent to F^i first order stochastically dominating F^{-i}. An implication from the first price auction literature is that bidder i's mean quantity demanded weakly exceeds bidder $-i$'s in equilibrium (Kirkegaard, 2009, Corollary 1). The stronger distributional ordering property of reverse hazard rate dominance can be stated as follows.

Definition 2 *(Reverse hazard rate dominance).*

$$F \succeq_{\text{rh}} G \iff \frac{d}{dx}\frac{F(x)}{G(x)} \geq 0,\ \forall x.$$

When F and G admit densities at x, this implies $f(x)/F(x) \geq g(x)/G(x)$. If $F^i(x)m$ is the mean residual supply curve that bidder i would present to bidder $-i$ if she were to bid her value for each unit, then $xf^i(x)/F^i(x)$ is the elasticity of that supply curve. The reverse hazard rate condition can then be interpreted as requiring that these elasticities are ordered. From Proposition 3.5 of Maskin and Riley (2000) we can therefore conclude that this ordering of elasticities is sufficient to order the bid curves of the bidders, meaning $F^i \succeq_{\text{rh}} F^{-i}$ implies $b^i(v) < b^{-i}(v)$ or $\mathbf{b}^i(\mathbf{v}) < \mathbf{b}^{-i}(\mathbf{v})$. This is the well-known "weakness leads to aggression" result.

Finally, we make one more connection to work on investment incentives in single unit auctions. In their Proposition 3 Arozamena and Cantillon (2004) show that if one bidder is given the opportunity to "upgrade" their type distribution ex ante by making it stronger with respect to hazard-rate dominance, the investment incentives are stronger in the second-price auction than in the first-price auction. Furthermore, their Proposition 4 shows that investment incentives are optimal in the second-price auction. Upgrading the distribution has a natural interpretation in our model. It is equivalent to a bidder in our model investing to increase her mean demand curve in such a way as to weakly increase the elasticity of the mean residual supply curve at every point. From the Arozamena and Cantillon (2004) results we get immediate comparisons of the investment incentives in the LAB auction to those in the Vickrey auction, which is the extension of the single-unit second price auction to this environment.

3.1.2. Equilibrium uniqueness

The arguments for uniqueness in the first price auction given in the literature typically follow a common set of intermediate steps.[28] First, one shows that the largest equilibrium bid (or smallest in the case of procurement) is the same for every bidder. Second, one defines a system of ordinary differential equations involving inverse bid functions. The equations in the system are shown to be necessary and sufficient for optimality and also to satisfy the Lipschitz condition at every bid but the lowest. The initial value problem starting from a particular highest bid therefore has a unique solution due to the fundamental theorem of ordinary differential equations (FTODE). Third, one shows that if $\bar{b}$ and $\widetilde{b}$ are two initial values with $\bar{b} < \widetilde{b}$ then the solutions to the initial value problem using $\bar{b}$ are greater than those to the problem using $\widetilde{b}$ at every interior b. Finally, one shows with an additional assumption about the problem at the lowest bid that the second and third results imply that there can only be one highest bid yielding a solution that is also an equilibrium.

To establish uniqueness of the LAB equilibrium among separating strategies in our model, we follow the first two steps above but then appeal to the uniqueness of the corresponding first price auction solution to complete the proof. We restrict attention to separating strategies, because our argument relies on the analysis of a system of differential equations that is only valid for separating strategies. Allowing the monotonicity constraint to bind for arbitrary bids leads to a system of equations that is substantially more difficult to analyze and, as we argue, less likely to describe observed bidding behavior.

The most important step in our argument is to establish a common upper bound on bids, independent of bidder and unit. As in single-unit auctions it is not crucial to identify the exact value of this bid, it is established only as an initial condition for the differential system. Unlike in single-unit auctions the commonality of maximum bids cannot be guaranteed from first principles. To see this, consider the two-bidder case in which $m_i = 3$ for both agents, and $m = 3$ units are for sale. Suppose that for either bidder, the maximum bid for unit 1 exceeds the maximum bid for units 2 and 3. Then a sufficiently high bid for unit 1 is never price-setting, and there is no disincentive to increasing the bid for unit 1 above the maximum bid for unit 2. Intuitively, when low-quantity bids are sufficiently high incentives in the last accepted bid auction are similar to those in a second price auction.

Arguing for common upper bounds proceeds in two steps. First, even if there is not a common upper bid both bidders' maximum bids for unit m must be weakly below their opponents' maximum bids for unit 1, because otherwise these bids occasionally determine the market clearing price and can be costlessly reduced. This implies that we can constrain attention to bids which

[28] We refer to Lebrun (2006) for a discussion of uniqueness results in the first price auction literature and the assumptions required to prove uniqueness. There is a unique equilibrium in the asymmetric first price auction under fairly general conditions, but as argued in Lebrun (2006) some prior proofs have relied on unjustified uses of l'Hôpital's rule.

cross opponent maximum bids. The second step in the argument is to show that just above this unit the density of bids is strictly positive (at the maximum bid). Since the density of order statistics disappears at the bounds of the support, this implies that inverse bid functions are infinitely steep at the maximum bid.

Lemma 3 *(Common upper bids). In any separating equilibrium of the LAB auction with Lipschitz continuous inverse bids, $\overline{b}_k^i = \overline{b}$ for both bidders $i \in \{1, 2\}$ and all units k.*

Proof. First, note that $\overline{b}_m^i \le \overline{b}_1^{-i}$ for any bidder i. Otherwise, $b_m^i > \overline{b}_1^{-i}$ with positive probability, and hence b_m^i sets the market price with positive probability. Then a small reduction in b_m^i will win the full market quantity with probability one while reducing the payment for this quantity, improving utility.

Then there is an agent i and a unit $k > 1$ such that $\overline{b}_k^i < \overline{b}_{k-1}^i$. Because $k > 1$ and inverse bids are Lipschitz continuous, $dH_k^i(\overline{b}_k^i) = 0$; because $\overline{b}_k^i < \overline{b}_{k-1}^i$, $H_k^i(\overline{b}_k^i) = 1 > H_{k-1}^i(\overline{b}_k^i)$. Bidder $-i$'s utility for unit $m - k + 1 \le m$ is

$$u_{m-k+1}^{-i}(b; v) = v_{m-k+1} H_k^i(b) - \left(H_k^i(b) - H_{k-1}^i(b)\right)(m-k+1)\, b$$
$$- (m-k+1)\, \mathbb{E}\left[b_{k-1}^i \middle| b_{k-1}^i \le b \right] H_{k-1}^i(b)$$
$$+ (m-k)\, \mathbb{E}\left[b_k^i \middle| b_k^i \le b \right] H_k^i(b).$$

This value is strictly decreasing in b for $b \in (\overline{b}_k^i, \overline{b}_{k-1}^i)$. It follows that $\overline{b}_{m-k+1}^{-i} \le \overline{b}_k^i$. Moreover,

$$\lim_{b \nearrow \overline{b}_k^i} \frac{d_+}{db} u_{m-k+1}^{-i}(b; v) = -\left(H_k^i(b) - H_{k-1}^{-i}(b)\right)(m-k+1) < 0.$$

Since equilibrium is separating, b_{m-k+1}^{-i} can depend only on v_{m-k+1}^{-i}. Then there is $\varepsilon > 0$ such that there is no v with $b_{m-k+1}^{-i}(v) > \overline{b}_k^i - \varepsilon$; hence $\overline{b}_k^i > \overline{b}_{m-k+1}^{-i}$. Since equilibrium is separating, bidder $-i$ can occasionally reduce her bid for unit $m - k + 1$, reducing her payment without affecting her allocation probability. This deviation is profitable, hence it is not possible that $\overline{b}_k^i < \overline{b}_{k-1}^i$. Then $\overline{b}_k^i = \overline{b}$ for both agents i and all units k. $\square$

We next describe the system of differential equations generated by the last accepted bid auction. As with first price auctions, the arguments are made simpler by writing the differential equations in terms of an unknown derivative with respect to a bid distribution (see, e.g., Lebrun (2006)). Recall that in this section we assume that bidders use separating strategies. This implies that $\varphi_k^i(b) \le \varphi_{k+1}^i(b)$ for all k and i. Consequently, the distribution of the kth bid of bidder i is $F_{(k)}^i(\varphi_k^i(b))$. Furthermore, bidder $-i$'s first order condition with respect to her $m - k + 1$th bid becomes

$$\left[\varphi_k^i\right]'(b)\, f_{(k)}\left(\varphi_k^i(b)\right)\left(v_{m-k+1}^{-i} - b\right)$$
$$- (m-k+1)\left(F_{(k)}^i\left(\varphi_k^i(b)\right) - F_{(k-1)}^i\left(\varphi_{k-1}^i(b)\right)\right) = 0.$$

We create a system of $2m$ differential equations out of the first order conditions for each bid by each bidder. Instead of writing the system in terms of unknown bid functions, we write it in terms of unknown bid distributions as follows.

Definition 3 *(LAB differential system).* Let $H_k^i \equiv F_{(k)}^i \circ \varphi_k^i$, $H_0^i(b) \equiv 0$, $\varphi_{m-k+1}^{-i} \equiv F_{(m-k+1)}^{-i,-1} \circ H_{m-k+1}^{-i}$, and $\overline{b} \in (0,1)$ be given. For $k \in \{1,\dots,m\}$, $i \in \{1,2\}$, and $b \in (0,\bar{b}]$, the differential system representing best response behavior is given by

$$\begin{aligned} \frac{d}{db} H_k^i(b) &= (m-k+1)\,\frac{H_k^i(b) - H_{k-1}^i(b)}{\varphi_{m-k+1}^{-i} - b}, \\ H_k^i\left(\overline{b}\right) &= 1. \end{aligned} \tag{4}$$

This initial value problem involves a system of $2m$ equations — m equations for each of 2 bidders — in $2m$ unknown functions, H_k^i. The following lemma establishes that an equilibrium of the LAB auction is necessarily a solution to this initial value problem.

Lemma 4. *Any separating equilibrium with Lipschitz continuous inverse bids must satisfy equation* (4).

Proof. See Appendix B. □

The derivation Lemma 4 is similar to the derivation of well-behavedness properties in first price auctions. Lipschitz continuity is necessary only to apply Lemma 3 and ensure that maximum bids are constant across all units. The differential system in Definition 3 is inapplicable only if opponent bids are nondifferentiable, which occurs only when bids have either kinks or constant intervals. Constant intervals are ruled out by the presumption of strict separation, so the differential system is necessary and sufficient for best response behavior as long as bid functions do not have kinks. Kinks in the bidder i's bid will imply either mass points in bidder $-i$'s bid, which is disallowed, or gaps in the support of bidder $-i$'s bid. If bidder i has a kink in her bid function at b while bidder $-i$ never bids near b, bidder i has a profitable deviation.[29]

Some care must be taken in our argument to deal with the possibility that profitable deviations are not available due to the monotonicity constraint binding. However, because payoffs are continuous in value, if the bid monotonicity constraint is binding for a bidder with value profile v, there is a nearby bidder with value profile $v' \neq v$ facing roughly the same incentives who submits the same monotonicity-constrained bid. This violates separation, and can therefore be excluded from our analysis.

Compare the problem in Definition 3 to the following corresponding one for the first price auction.

Definition 4 *(FPA differential system).* Let $H^i \equiv F^i \circ \varphi^i$ and $\overline{b} \in (0,1)$ be given. For $i \in \{1,2\}$ the differential system representing best response behavior is given by

$$\begin{aligned} \frac{d}{db} H^i(b) &= \frac{H^i(b)}{\varphi^{-i} - b} \\ H^i\left(\overline{b}\right) &= 1. \end{aligned} \tag{5}$$

[29] It is worth noting that ruling out flats in the bid function is not sufficient to imply Lipschitz continuity of the inverse bid function. For example, it could be that $db_k^i/dv(v) = 0$ for some v. Requiring that the bid for any unit is independent of the unit for which it is submitted is sufficient to avoid this technicality, but this assumption begs the question of uniqueness.

For an arbitrary $\overline{b}$, because (5) satisfies the Lipschitz condition for all $b \in (0, \overline{b}]$, the FTODE implies there is a unique solution to the initial value problem in Definition 4. Furthermore, when there is a unique equilibrium in the first price auction, a single such $\overline{b}$ yields a solution that also satisfies the boundary condition $H^i(\underline{b}) = F^i(\varphi^i(\underline{b})) = 0$, where $\underline{b}$ is the lowest equilibrium bid.

Since the system in (4) also satisfies the Lipschitz condition for all $b \in (0, \overline{b}]$, the FTODE implies that there is a unique solution to the problem in Definition 3, given $\overline{b}$. But these two solutions must coincide: when (φ^1, φ^2) is a solution to the first price auction problem, the unique solution to the LAB initial value problem can be found by setting $\varphi^i_k = \varphi^i$ for all k and i. The final step is to observe that while Proposition 1 implies that equilibrium value of $\overline{b}$ generates equilibrium solutions to both problems, a different $\overline{b}$ would generate a solution that is not an equilibrium of the first price auction (by uniqueness) and cannot be an equilibrium of the LAB auction.

Proposition 2. *Consider the LAB auction with $n = 2$ bidders, each with demand for $m_i = m$ units. Suppose that bidder i's values are given by m_i ordered draws from the distribution F^i. If there is a unique equilibrium in the first price auction with $n = 2$ bidders drawing values from F^1, F^2, then there is a unique equilibrium of the LAB auction in which the bidders use separating strategies with Lipschitz continuous inverse bids.*

In Appendix D we prove equilibrium uniqueness when there are n symmetric bidders and $m = 2$ units are for sale. The approach is mathematically similar but the methods must be significantly altered. In more general environments, separating equilibria can (and in some cases must) sustain quantity-dependent maximum bids, invalidating application of Lemma 3.

3.2. Symmetric bidders in a balanced market

Under market balance equilibrium bids in the LAB auction have a clear tie to equilibrium bids in the single-unit first price auction, as shown in Proposition 1. In the case of bidder symmetry, this result extends to the general n-bidder case, resulting in a familiar functional form for equilibrium bids.

Equation (2) shows that each bidder's incentives depend only on the difference of two order statistics distributions, which depends on k in a straightforward way: at the optimum, a bid for a particular unit is determined only by the probability that this is exactly the bidder's marginal unit. These are exactly the incentives in a standard single-unit first price auction, leading to Proposition 3.

Proposition 3 *(Equilibrium in symmetric LAB). If marginal values for each bidder are the order statistics from independent draws from F the equilibrium bids for bidder i in the LAB auction are*

$$\mathbf{b}^i(\mathbf{v}_i) = \left(\frac{1}{F\left(v^i_k\right)} \int_0^{v^i_k} x f(x)\, dx \right)_{k \in \{1,..,m_i\}}.$$

Proof. Applying facts about order statistics, the costs in equation (2) are given by

$$\left(H^{-i}_{m-k+1}(b) - H^{-i}_{m-k}(b)\right) k = \binom{m}{m-k} (1 - F(\varphi(b)))^{m-k} F(\varphi(b))^k k.$$

The benefits in equation (2) are given by

$$
\begin{aligned}
&(v_k - b)\, dH^{-i}_{m-k+1}(b) \\
&\quad = (v_k - b_k)\binom{m}{m-k} F(\varphi(b))^{k-1}(1 - F(\varphi(b)))^{m-k} k f(\varphi(b))\, d\varphi(b).
\end{aligned}
$$

Then equation (2) becomes

$$(v_k - b_k)\, f(\varphi(b))\, d\varphi(b) = F(\varphi(b)).$$

It is clear that the candidate equilibrium bid, $b(v^i_k)$, satisfies the kth first order condition. Standard arguments establish that the partial derivative is negative (positive) for $b(v')$ when $v' > v^i_k$ ($v' < v^i_k$), which also means that the objective must be lower at the end points, $b(v^i_{k-1})$ and $b(v^i_{k+1})$, given by the monotonicity constraint. $\square$

Several comments are in order. First, this bid form depends on market balance. When market balance is not satisfied the k-fold increase in payment is balanced against a marginal tie that occurs in $\tau_k \neq k$ ways, and the factors no longer cancel. This also shows that market balance does not generalize to other algebraic conditions (as symmetric single-unit first price auction bids generalize to "highest of $n-1$ draws from F, below v"). What is essential for equilibrium is not that $k = \tau_k$, but that k/τ_k is constant. Letting $k' = k+1$, when market balance is not satisfied $k'/\tau_{k'} = (k+1)/(\tau_k+1) \neq k/\tau_k$. Then equilibrium bid functions cannot be independent of the unit for which they are submitted, and bids are no longer distributed as order statistics from a common distribution.

Second, this shows why other multi-unit auction formats fail to generate the same incentives as single-unit first price auctions. In a PAB auction the bid is paid one time over. Because bids are distributed as order statistics when values are distributed as order statistic only if the number of times paid is in constant ratio to the number of opponent bids beaten, the order statistic model arises only when at most one unit is demanded. Trivially, a multi-unit PAB auction generates the same incentives as a single-unit first price auction only if it is a multi-unit auction for a single unit. In a FRB uniform price auction this ratio argument does not apply, since the probabilities are not of the same order.

Finally, the equilibrium in Proposition 3 is efficient, and hence standard arguments imply that the expected payment should be equal to the Vickrey payment. To see this, let $Y_{(k)}$ denote the kth order statistic from m draws from F. Consider the event that $b(v^i_k)$ is the last accepted bid (i.e., $Y_{(m-k)} \geq v^i_k \geq Y_{(m-k+1)}$). In this event the bidder pays $kb(v^i_k)$, which is the expected payment made for k units in a Vickrey auction conditional on this event because

$$
\begin{aligned}
&\sum_{j=1}^{k} \mathbb{E}\left[Y_{(m-k+j)} \,\middle|\, Y_{(m-k)} \geq v^i_k \geq Y_{(m-k+1)} \right] \\
&\quad = \sum_{j=1}^{k} \mathbb{E}\left[Y_{(j:k)} \,\middle|\, v^i_k \geq Y_{(1:k)} \right] = k\mathbb{E}\left[Y \,\middle|\, v^i_k \geq Y \right] = kb\left(v^i_k\right), \qquad (6)
\end{aligned}
$$

where the notation $Y_{(j:k)}$ denotes the jth highest value out of k independent draws from F. To understand the first equality, observe that conditional on the event $Y_{(m-k)} > v^i_k > Y_{(m-k+1)}$ the first $m-k$ random variables provide no additional information about the last k random variables, so the expectation reduces to one involving just the last k. Recall that in the Vickrey auction a bidder who wins k units in this environment would be required to pay the sum of the k rejected bids made by the opponents. In other words, the bids in this equilibrium are set so that the expected payment equals the expected Vickrey auction payment conditional on the event that the bid determines the payment.

4. Comparison to other multi-unit auctions

The previous section shows that there is a close connection between the equilibrium of the first price auction and that of the LAB auction. In this section we discuss properties of these equilibria, focusing on separation and information transmission, and use a large market approach to compare our results to much of the literature on uniform price auctions. Uniqueness and separation lie at the heart of the practical value of the LAB auction model: separation ensures that, in equilibrium, per-unit first order conditions are satisfied with equality, while uniqueness ensures that bidders in practice will behave comparably across auction implementations. Our comparison with known results from the uniform price auction literature suggests that the understanding that uniform price auctions generate relatively steep bid curves may rely on the lack of private information in existing models.

We contrast the properties of the LAB auction discussed in the previous sections with those of two other common multi-unit pricing rules, the PAB auction and the FRB uniform price auction. Propositions 1 and 3 establish the existence of separating equilibrium in the LAB auction when market balance is satisfied; Proposition 11 shows the same in a model where market balance is not satisfied. Separation is demonstrated by direct computation of equilibrium strategies. We show in this section that neither the FRB nor the PAB auctions admit separating equilibria, whether or not market balance is satisfied. A direct implication is that these auctions cannot generalize the equilibrium incentives of single-unit first price auctions to a multi-unit context. Bids in the FRB uniform price auction cannot generalize single-unit first price auction incentives because there is pooling on zero bids, which cannot occur in the single-unit context. These same incentives give rise to zero-revenue equilibria in the FRB auction (see, e.g., Milgrom (2004)), behavior which is not supportable in the LAB auction. Bids in the PAB auction cannot generalize single-unit first price auction incentives because the monotonicity constraint must bind, and bids cannot be independently determined unit-by-unit. The existence of a separating equilibrium demonstrates the superior informational properties of the LAB auction.[30]

4.1. Separation

First price auctions do not typically admit equilibria in which participants with different values pool on identical bids. The intuition is straightforward and applies to many other single-unit auction contexts: the presence of a mass point in one bidder's distribution of bids implies a discontinuity in the density of the distribution of the winning bid, generating disincentives to opponents submitting bids either just above or just below the mass point. The lack of one opponent's bids in the neighborhood of the mass point generates a feedback effect by which other

[30] McAdams (2008) shows that bidder values may not be point-identified by observed bids in pay as bid and FRB uniform price auctions. In the case of the discriminatory auction the underlying intuition is similar to ours: conditional on a distribution of opponent bids, flat bids may be generated by any of a set of value profiles. Our results show that, in these auctions, all equilibrium bid profiles are at least partially pooling. Then bidder values cannot be point-identified from observed equilibrium bids.

opponents are even more disincentivized to bid in this neighborhood, and this behavior cannot be sustained in equilibrium. Any monotone strategy that does not generate mass points in the bid distribution must be such that all types submit different bids, and therefore equilibrium is separating.

However in multi-unit auctions there is the potential for probability-zero pooling behavior, in which multiple types submit the same bid but any given bid is submitted with probability zero. For example, a bidder who demands two units could bid her average value for both units. Then the distribution of her bids is exactly the distribution of underlying types, which is massless, but any given bid is submitted by more than one type of bidder. This effect arises when bids for distinct units must be determined simultaneously, for example in the presence of a binding monotonicity constraint. Simultaneous bid determination is unique to multi-unit auctions and cannot occur in single-unit auctions, where bids are one-dimensional. Because of this possibility, in multi-unit auctions pooling cannot be ruled out from economic first principles, and separation becomes an interesting question in its own right.

Propositions 1 and 3 show that there is a separating equilibrium in LAB, a natural distinction between it and the other two auction formats.

Corollary 1. *Suppose there are n symmetric bidders, or $n = 2$ bidders with potentially asymmetric value distributions, with demands satisfying market balance. There exists an equilibrium of the LAB auction in which bids are strictly separating.*

By way of contrasting behavior in the FRB and PAB auctions with the strict separation we have shown to be possible in the LAB auction, we introduce the notion of partial pooling.

Definition 5 *(Partial pooling).* A bid function $\mathbf{b}^i$ exhibits *partial pooling* if the inverse bid correspondence is multi-valued with positive probability; that is,

$$\Pr\left(v \in \left\{v' : \#\varphi^i\left(\mathbf{b}^i\left(v'\right)\right) > 1\right\}\right) > 0.$$

A bid function exhibits partial pooling if there is a positive-probability set (of bids) on which values cannot be uniquely identified. There is a wedge between strict separation and partial pooling: inverse bids might be multi-valued with zero probability. We are concerned with issues of information confounding, and in particular in situations in which information is obfuscated in equilibrium. If equilibrium bids are non-separating with probability zero, equilibrium is essentially separating, and this distinction is not meaningful.

Our definition of partial pooling is structured to capture two separate pooling effects. In the FRB auction, truthful bidding for the first unit is a weakly dominant strategy. However, we show that there is a range of last-unit valuations such that a bid of zero strictly dominates all others; this occurs because residual competition comes from opponents' small quantities, for which bid distributions are relatively strong. Increasing the bid for the final unit has little marginal effect on the probability of winning the unit but a comparatively strong marginal effect on the expected cost paid for all $m_i - 1$ units, conditional on their being won. In an equilibrium with truthful bids for the first unit, the probability of witnessing any particular bid is zero even though the probability of witnessing a zero bid for an agent's final unit is strictly positive; partial pooling captures this positive-probability noninvertibility.

Partial pooling also captures the information confounding we observe in the PAB auction. In PAB, the bidder is facing increasingly aggressive competition as she considers her bid for higher units: her bid for higher units is against her opponents' bids for lower units. We show that there is generally an incentive for the idealized bid for the first unit — the optimal bid when the monotonicity constraint is ignored — to be below the idealized bid for the second unit, violating the bid monotonicity constraint. This implies that, for certain value profiles, bids will be flat for small quantities. Continuity of utility in value implies that this same flat will be realized for nearby value profiles — if, for example, the value for the first unit falls while the value for the second unit rises — and thus upon witnessing a particular flat bid the bidder's value profile cannot be perfectly inverted. Again, this happens in spite of no particular bid profile being submitted with strictly positive probability.

Aside from implications for tractability, information revelation is directly related to efficiency. An efficient mechanism must allocate units to the agents with the highest values. When information is confounded, this is not possible: efficiency entails the mechanism designer knowing which agents have the highest values for the m available units, and standard identification arguments imply that if this is possible, bids must be separating. We thus contrast the FRB and PAB auctions, in which all equilibria exhibit partial pooling and are thus inefficient, with the LAB auction, which we have shown to admit a separable and efficient equilibrium without pooling.

Remark 1. Any pure strategy equilibrium can be transformed into a monotone pure strategy equilibrium without affecting agents' incentives or payoffs. We therefore restrict attention to equilibria in monotone pure strategies.

Remark 1 is familiar from other auction contexts. Because bidders with higher values are at least as willing to submit marginally higher bids as bidders with lower values, if bids are nonmonotone in value the bidder with the higher value must be at least indifferent between the pointwise maximum of the two bids and the bid she is submitting. Then bidders with these two value profiles can "swap" their bids for the pointwise maximum and minimum, and their utilities will be (at worst) unaffected.

Under separation and monotonicity, the intuitive notion that bids are optimized unit-by-unit is quickly formalized.

Lemma 5 *(Separable bids in separating equilibrium). In a monotone strictly separating equilibrium of the LAB, FRB, or PAB auction models, bidder i's equilibrium bid function can be written as*

$$b^i(v) = \left(b_1^i(v_1), \ldots, b_{m_i}^i\left(v_{m_i}\right)\right).$$

Proof. See Appendix C. □

Taken together, the above results imply that either equilibrium bids can be analyzed independently, unit-by-unit, or equilibrium exhibits partial pooling. Following the definition of partial pooling, this implies that when bids cannot be analyzed independently equilibrium outcomes must be inefficient, and information is not fully revealed. When comparing the PAB and FRB auctions to the LAB auction, these results allow us to analyze the revelation question dimension-by-dimension and, from these dimensional analyses, to build contradictions which expose the prevalence of partial pooling.

4.2. Partial pooling in the FRB auction

In the FRB auction bids for large quantities are disproportionately unprofitable. A small increase in bid for a large quantity implies that, when this bid is supra-marginal, this increase is paid for each unit won. Because a bidder competes for large quantities against her opponents' small quantities, not only is there an outsized cost associated with increasing this bid but there is also only a relatively small increase in the probability of winning. These incentives balance in favor of a mass point at a bid of zero.

To eliminate pathological cases, we define the notion of a well-behaved equilibrium.

Definition 6 *(Well-behaved bids).* A bid function b_k^i is *well-behaved* if $d_+^t b_k^i / dv^t$ is bounded on $(0, 1)$, for all finite t. The bid profile $((b^i)_{k=1}^{m_i})_{i=1}^n$ is well-behaved if b_k^i is well-behaved for all agents i and all units k.

Lemma 6 *(Partial pooling in FRB). If aggregate supply is $m \geq 2$ and each bidder demands $m_i \geq 2$ units, all well-behaved equilibria of the FRB auction exhibit partial pooling.*

Proof. See Appendix C. □

Lemma 6 is explicitly observed in the two-unit example of Section 1.1.2. The general principles at work in the order statistic model are readily observed when $m < \sum_{i=1}^n m_i$.[31] Increasing bidder i's bid for unit m_i has two effects: first, it increases the probability that she wins unit m_i. Second, it increases her expected payment conditional on winning $m_i - 1$ units, because occasionally when she wins $m_i - 1$ units her bid on her m_ith unit sets the market clearing price. In the order statistic model, the event that she wins $m_i - 1$ units has probability an order (in terms of the probability distribution) greater than the event that she wins m_i units, so there is an outsized cost associated with increasing her bid.[32] It follows that when her value for unit m_i is relatively low, she will not submit a strictly positive bid for this unit.

It is worth clarifying the role of well-behavedness in Lemma 6. If the necessary limit (in the proof) does not exist, the Lemma is automatically satisfied: the limit will fail to exist only when the ratio can be discretely positive for b arbitrarily close to 0. This alone is sufficient to indicate that pooling at 0 is advantageous. Thus well-behavedness supports bid density, and provides that $d_+^{(t+1)} H_{m-k+1}^{-i}(0)$ is finite at the smallest t for which it is nonzero. We do not know of an economic interpretation of this derivative being infinite while all lower derivatives are zero, but nor can it be ruled out of hand.

Remark 2. With $n = 2$ bidders and $m \geq 2$ units, all equilibria of the FRB auction in weakly dominant strategies exhibit partial pooling. Because truthful reporting for the first unit is weakly dominant, $d_+^{(t)} H_1^{-i}(0)$ is finite for the lowest t at which it is nonzero, implying that Lemma 6 can be applied directly.

In light of Remark 2 and the preceding discussion, we obtain the following proposition.

---

[31] When $m \geq \sum_{i=1}^n m_i$ pooling arises due to market degeneracy: there is weak excess supply in the market, and there is no incentive to submit any positive bid.

[32] The sharpness of this intuition follows from the order statistic model. However, pooling on zero bids in the FRB auction can be observed in other informational contexts; see, e.g., Ausubel et al. (2014).

Proposition 4 *(Inefficient equilibrium in FRB). All equilibria of the FRB auction satisfy one (or more) of the following two properties:*

i. *Equilibrium is inefficient.*
ii. *For all agents i and all units k, $d_+^{(t)} H_{m-k+1}^{-i}(0)$ is infinite at the lowest t at which it is nonzero.*

As discussed above, efficient equilibria must be separating. Then in light of Lemma 6 equilibrium cannot be well-behaved and separating simultaneously. In a natural sense, equilibria that would be expected to be observed must be inefficient.

Point ii. of Proposition 4 appears to be a technicality, and we do not know of any equilibrium constructions satisfying point ii. Importantly, equilibrium is inefficient unless all bidders employ the same bidding function for all units. If the underlying value distributions are C^∞ on $[0, 1]$, for point i. to be unsatisfied while point ii. is satisfied there must be $\tau > 0$ such that $d_+^{(t)} b(0) = 0$ for all $t < \tau$, and $d_+^{(\tau)} b(0) = \infty$. As mentioned above we cannot rule this out of hand, but this places a tight restriction on strategic behavior for the FRB auction to be efficient.

Corollary 2 *(Inefficient equilibrium in FRB (two bidders)). When underlying value distributions F^i are C^∞ and there are $n = 2$ bidders, all equilibria of the FRB auction are inefficient.*

When there are only two bidders, $H_{m-k+1}^{-i} = F_{(m-k+1)} \circ \varphi_{m-k+1}^{-i}$. If equilibrium is efficient, bid functions are independent of agent and unit, and there is φ such that $\varphi_{m-k+1}^{-i} \equiv \varphi$. Furthermore point ii. of Proposition 4 must be satisfied. If this holds for all units, bids are locally dense at zero for all units. When this is the case bids for unit $k = 1$ are truthful when values are sufficiently small, $\varphi(v) = v$ when v is small, and $\varphi' = 1$ and $\varphi'' = 0$. If value distributions are suitably well-behaved point ii. is contradicted. This logic does not immediately extend to the case of $n > 2$ bidders because in general there is no unit uniquely in opposition to unit k — there is instead an iso-residual demand curve. Then while first-unit truthfulness is still satisfied, it no longer implies that bid distributions have bounded derivatives.

4.3. Partial pooling in the PAB auction

As the bid-for quantity increases, the marginal distribution of opponent values shifts upward; the relative lack of competition for small quantities implies partial pooling in the PAB auction. In the case of two bidders a bidder will win unit 1 if and only if her opponent does not win unit m; similarly the bidder will win unit 2 if and only if her opponent does not win unit $m - 1$. Since her opponent's marginal distribution of values for unit $m - 1$ dominates the distribution of values for unit m, the bidder faces less competition for unit 1 than she does for unit 2. Ideally, given a value $v_1 = v, v_2 = v$ she would bid less for unit 1 than for unit 2, violating the monotonicity constraint.[33]

When the monotonicity constraint is binding, bids cannot be written as profiles of independent bids across the units the bidder demands. Intuition suggests that partial pooling is present: if the bid $b_1 = b, b_2 = b$ is observed, it cannot be known whether this bid has resulted from individual unconstrained bids, or bids that pass through the monotonicity constraint. Then values cannot be inverted out from observed bids, and there is some degree of pooling present in equilibrium.

[33] This has been explored in a divisible-good context by Woodward (2016).

We now turn attention to *continuous-bid* equilibria, in which bids are continuous functions of value. In the two-bidder case standard arguments suffice to rule out discontinuities in equilibrium bid functions, but in general this is less clear.[34] Nonetheless, we are able to show that equilibria in continuous bids exhibit partial pooling. Since partial pooling implies inefficiency, as do discontinuous bids, it follows that all equilibria in the PAB auction are inefficient. Further, inasmuch as partial pooling makes equilibrium computation more difficult, and discontinuous equilibria also present computational challenges, these results can be taken as suggesting a general intractability of PAB auction equilibria.[35]

To begin our analysis, we define the notion of a maximal bidder. Lemma 7 implies that at least two maximal bidders exist.

Definition 7 *(Maximal bidder).* Bidder i is a *maximal bidder* if for all k, $\bar{b}_k^i = \bar{b}$.

Lemma 7 *(Equal upper bounds). Let $\bar{b}$ be the maximum bid submitted for any unit, and let $\bar{b}_k^i$ be bidder i's maximum bid for unit k. There are two bidders, i and j, such that $\bar{b}_k^i = \bar{b}_{k'}^j$ for all units k and k'.*

Proof. See Appendix C. □

Maximal bidders feature prominently in our analysis of partial pooling because they are always in competition with one another. The existence of at least two maximal bidders ensures that when bids are relatively high, the model's first order conditions must be satisfied with equality. As we show in Lemma 8, this implies a mass point at the upper bound of the distribution of bids, for at least one agent-unit tuple.

Lemma 8 *(Monotonicity of distributional differences). Let i be a maximal bidder, and let $\delta_k(b) = H_{m-k+1}^{-i}(b) - H_{m-k}^{-i}(b)$. In any strictly-separating, continuous-bid equilibrium of the PAB auction, $\lim_{b \nearrow \bar{b}} \delta_k(b) > 0$ for all $k \in \{1, \ldots, m-1\}$.*

Proof. See Appendix C. □

Corollary 3 *(Partial pooling in PAB). Continuous-bid equilibria in the PAB auction exhibit partial pooling.*

Remark 3. With $n = 2$ bidders, all equilibria of the PAB auction exhibit partial pooling. When there are only two bidders in the model, the problem is analogous to a set of simultaneous two-bidder asymmetric auctions. Since the support of valuations is convex, standard results imply that bids are continuous in value, satisfying the antecedent of Lemma 3.

---

[34] This is observed throughout the single-unit auction literature; with multiple units, the problem is exacerbated. For example, if each bidder has positive value for all m units ($m_i = m$ for all i), then when determining the bid for her $m-2$nd unit a bidder must consider not only the possibility that any of her opponents receives 2 units and all the others receive 0 units (this is analogous to the single-unit case), but also to the possibility that any combination of two opponents each receives 1 unit while all others receive 0 units. This iso-allocation set makes standard no-gaps arguments inapplicable.

[35] This is in line with known results about multi-unit auctions, including Hortaçsu and Kastl (2012).

Lemma 7 establishes the existence of (at least) two bidders who submit the same maximum bid for any unit. Compared to unit $\hat{k}$, there is a relatively small probability of having a high value for unit $\hat{k}+1$; however, *conditional* on having a high value for both units the bidder sees higher returns to increasing his bid for unit $\hat{k}+1$ than to increasing his bid for unit $\hat{k}$. This is in line with results in first-price auctions showing that distributional weakness leads to bid aggression. Since there is no value to bidding above the upper bound of the support of bids $\bar{b}$, strong incentives to bid on unit $\hat{k}+1$ induce a mass point at $\bar{b}$; this is our Lemma 8.

While on its face Lemma 8 appears to directly imply partial pooling — multiple types submit identical bids on single dimensions — this is not a complete characterization of partial pooling in the PAB auction. As is standard in other auction contexts, mass points in the bid distribution are not possible in equilibrium (above its lower bound). The construction in Lemma 8 contradicts this logic. Thus any separating equilibrium would have mass points at the upper bound of the bid distribution, which cannot occur in equilibrium. It follows that in any pure strategy equilibrium, partial pooling arises even though no bid occurs with strictly positive probability.

Partial pooling in the PAB auction arises due to the monotonicity constraint binding. Corollary 3 establishes this in some generality, but the intuition is most straightforward in the 2×2 balanced market case. Bidder i's bid for unit 1 is determined by her competition against bidder $-i$'s second-unit bid, and bidder i's bid for unit 2 is determined by her competition against bidder $-i$'s first-unit bid. In the order statistic model, the distribution of values for the first unit first order stochastically dominates the distribution of values for the second unit. If bids are determined independent of the monotonicity constraint, the "weakness leads to aggression" result from single-unit first price auctions implies that bidder i submits a more aggressive bid function for her second unit than for her first. When her values for these units are roughly equal, she will want to bid higher on her second unit than on her first, causing her monotonicity constraint to bind.

Remark 4. Under market balance the LAB auction admits a strictly-separating equilibrium, while equilibria in the FRB and PAB auctions exhibit partial pooling.[36] In the FRB auction, pooling arises due to low-value bidders submitting zero bids. In the PAB auction, pooling arises due to the monotonicity constraint binding when $v_{k+1} \approx v_k$.

Proposition 5 *(Inefficient equilibrium in PAB). All equilibria of the pay-as-bid auction are inefficient.*

Proof. In equilibrium, bid functions are either continuous or have a discontinuity somewhere. If they are continuous, equilibrium exhibits partial pooling (Lemma 3). Because the closure of the equilibrium market clearing price set must be convex, where bids are discontinuous it must be that higher-value agents sometimes lose a unit to lower-value agents, implying inefficiency. In either case, outcomes are inefficient. □

4.4. Zero-revenue equilibria and multiplicity

It has been noted (see, e.g., Engelbrecht-Wiggans and Kahn (1998), and Ausubel et al. (2014)) that the FRB auction frequently admits equilibria with arbitrarily small seller revenue. As implied

[36] Continuous-bid equilibria in the PAB auction exhibit partial pooling. Well-behaved equilibria in the FRB auction exhibit partial pooling.

by the proof of Lemma 6, all well-behaved equilibria in the FRB auction yield zero revenue with positive probability. In this section we focus on *zero-revenue* equilibria, which yield zero revenue with probability one. We show that the LAB auction does not admit any such equilibria.[37]

Proposition 6 *(Zero-revenue equilibrium in FRB). The FRB auction always admits zero-revenue equilibria. The LAB auction never admits zero-revenue equilibria.*

Proof. Construction of zero-revenue equilibria in a multi-unit setting is similar to that in a single-unit setting[38]: take numbers $(\tilde{m}_i) \in \mathbb{N}_0^n$ such that $\sum_{i=1}^n \tilde{m}_i = m$. Pick $\bar{s} \geq 1$ and let bidder i submit the bid

$$b^i(q; v) = \begin{cases} \bar{s} & \text{if } q \leq \tilde{m}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then the equilibrium price is always zero, independent of the bidders' private information; moreover, to win a greater quantity agent i must bid $\bar{s}$ for unit $\tilde{m}_i + 1$, obtaining weakly negative gross utility on this unit and incurring an additional payment of $\tilde{m}_i \bar{s}$. This is never utility-improving, hence these bid functions represent an equilibrium.

It is straightforward to show, by contradiction, that the LAB auction does not admit zero-revenue equilibria. Letting $q^i(v^i, v^{-i})$ be the equilibrium quantity allocation of agent i given value profiles v^i and v^{-i}, it is without loss of generality to assume that $q^i(v^i, v^{-i}) < m$ with positive probability.[39] Note that for almost all v^{-i} such that $q^i(v^i, v^{-i}) < m$, $b^i(q^i(v^i, v^{-i}) + 1; s_i) = 0$; furthermore, $b^{-i}(m - q^i(v^i, v^{-i}) + 1; v^{-i}) = 0$. Then by increasing her bid for units $q > q^i(v^i, v^{-i})$ to $\varepsilon > 0$, bidder i will incur an additional cost of at most $m\varepsilon$ but will win unit $q^i(v^i, v^{-i}) + 1$ with discretely positive probability. For ε sufficiently small this deviation is profitable, hence there is no low-revenue equilibrium. □

The FRB equilibrium constructed in Proposition 6 generates zero revenue with certainty. Furthermore, the proof of Proposition 4 implies that *all* well-behaved equilibria of the FRB auction yield zero revenue with strictly positive probability. As long as there is no mass point at the bottom of the value distribution, $F^i(0) = 0$, no equilibrium of the LAB auction can generate zero revenue with strictly positive probability.

Remark 5. In their divisible-good model, Back and Zender (1993) identify equilibria of the LAB auction that yield arbitrarily low revenue. The contrast between our results on the LAB auction, which rule out low-revenue equilibria, and their results is explained by an important distinction between divisible and indivisible goods. Intuitively, with indivisible goods in an LAB auction zero revenue requires that some bidder wins a unit with a bid of zero. This cannot happen with positive probability in equilibrium, because then a unit would be available to other bidders for an arbitrarily small price. In a divisible good model, the "last unit" purchased at the lowest price is itself arbitrarily small and hence its allocation does not influence payoffs. Provided residual supply is sufficiently steep, no bidder has an interest in increasing her bid. One interpretation of

[37] Our previous results are robust to the introduction of a strictly positive reserve price. In the (multi-unit) Vickrey auction a strictly positive reserve price — even infinitesimally small — is sufficient to eliminate zero-revenue equilibria; see, e.g., Blume and Heidhues (2004), and Blume et al. (2009).

[38] See, e.g., Milgrom (2004), pages 262–264.

[39] This follows from market clearing and the fact that we can focus on any particular agent.

Proposition 6 is that the "steep residual supply" logic behind the equilibria identified in Back and Zender (1993) carries over into the discrete setting only if the pricing rule is FRB. In fact, the idea behind the equilibrium construction in Back and Zender (1993) is essentially the same as that behind our construction of zero revenue equilibria in the FRB auction.

4.5. Large supply

Existing work providing tractable equilibria in uniform price auctions typically takes a divisible-good approach (cf. Klemperer and Meyer (1989); Back and Zender (1993); Ausubel et al. (2014)). As a final comparison of our results with the literature we consider equilibrium in our model in the large supply limit, assuming market balance holds along the limiting path. For simplicity we model this as each bidder having demand 1 while the good becomes divisible, so that bidders' allocations are given in $1/m$ units. Equivalently, we consider average per-unit utility as the quantity demanded becomes large.

Definition 8 *(m-divisible model).* In the *m-divisible model* with market balance, n bidders compete for shares of $Q = n - 1$ units of aggregate supply. Bidders' values are m-dimensional vectors distributed according to the order statistic model. The LAB auction is run as in the standard model, with the exception that the allocation q_i is such that $mq_i \in \mathbb{N}_0$. Bidder utility is given by

$$u^i\left(q_i, p; v^i\right) = \sum_{k=1}^{mq_i} \frac{1}{m} v_k^i - pq_i.$$

In the large supply limit underlying supply becomes divisible, $m \nearrow \infty$. In this limit bidders' value distributions retain no private information. Letting $qm \in \mathbb{N}$, when the number of units is large the distribution of values for unit $k = qm$ is approximately normal,

$$v_{k,m} \dot{\sim} N\left(F^{-1}(1-q), \frac{(1-q)q}{f\left(F^{-1}(1-q)\right)^2 m}\right).$$

The variance of the distribution goes to zero in m, thus in the large supply limit bidder i's value for unit $k = qm$ is deterministic and equal to the inverse underlying value distribution, $F^{-1}(1 - q)$. Since there is no private information bidders are competing in a full information auction with known demand curves; the case of an underlying uniform distribution is illustrated in Fig. 5. We refer to these inverse value distributions as asymptotic demand, tying full-information auctions to our private information model.

In the multi-unit model with private information, bids are vectors and a bid function is a map from m_i-dimensional private information to an m_i-dimensional bid. Because private information disappears in the large-supply limit it is no longer necessary to represent the bid for a particular quantity as a function of the bidder's private information. We simplify notation and use a bid curve to represent a map from quantities into bids, as is standard in divisible-good analyses.

Because market balance is satisfied along the limiting path, the bid profile in Proposition 3 is an equilibrium everywhere along the limiting path. Large supply equilibrium bid curves follow immediately from Proposition 3.

Proposition 7 *(Large supply equilibrium). In symmetric large supply equilibrium, bid curves are given by*

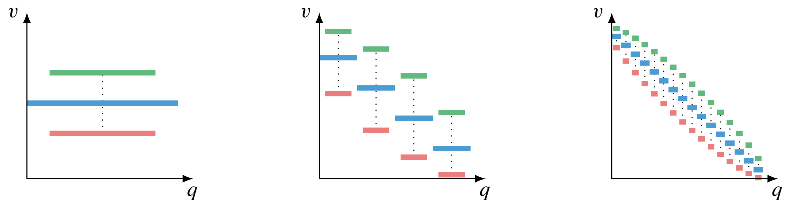

Fig. 5. Mean value and 10th and 90th percentiles as the number of units changes, $m \in \{1, 4, 16\}$, assuming a uniform distribution of values. As m becomes large the percentile ranges shrink, and private information disappears.

$$b^i(q) = \mathbb{E}_{v \sim F}\left[v | v \leq F^{-1}(1-q)\right].$$

Proposition 7 states that bids take the same form as in Proposition 3, in which bids are the expected value of a random draw v_{-i} conditional on it being below v. In the large supply limit the direct dependence on value is eliminated, since values are deterministic functions of quantity.

We have shown a clear distinction between LAB and FRB pricing in every multi-unit auction with finite aggregate supply, in terms of the information generated by the auction. However when supply is infinite — equivalently, when goods are divisible — and bids are continuous there is no distinction between the last accepted bid (the supremum of market clearing prices) and the first rejected bid (the infimum of market clearing prices). Where we assume deterministic (and inelastic) supply with private information, existing work (Klemperer and Meyer, 1989; Ausubel et al., 2014) has analyzed uniform price auctions with random supply and full information. We refer to these models as random supply models, and our model as a private information model.

Equilibrium in random supply models is ex post: no ex post allocation and transfer can be improved upon by a deviation. Then these equilibria are also equilibria when supply is deterministic. Because our large supply limit has no private information (as in random supply models), and existing approaches to uniform price auctions are equally valid when supply is deterministic (as in our private information model), we use our large supply model to compare equilibria of the two uniform-pricing rules.

In this setting all pure strategy equilibria yield a deterministic market clearing price $p^\star$. Klemperer and Meyer (1989)[40] show that any linear bid function b is supportable in equilibrium, so long as

$$b'\left(p^\star\right) = -\left(F^{-1}\left(\frac{1}{n}\right) - p^\star\right)n. \tag{7}$$

The bid curves in Proposition 7 have the following properties.

Corollary 4 *(Properties of large supply equilibrium). Symmetric large supply equilibrium bids are such that*

$$p^\star = \mathbb{E}\left[v \middle| v \leq F^{-1}\left(\frac{1}{n}\right)\right],$$
$$b'\left(p^\star\right) = np^\star - nF^{-1}\left(\frac{1}{n}\right).$$

40 Equation (7) is a translation of the results in Klemperer and Meyer (1989) to our model.

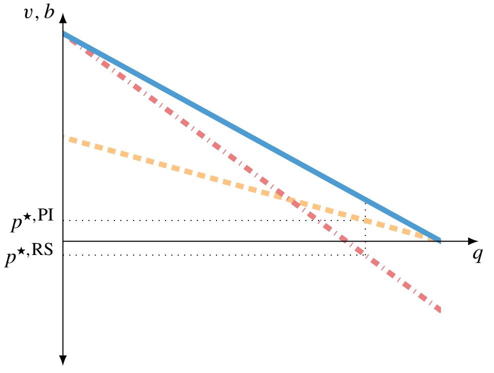

Fig. 6. Equilibrium bids are steeper in the linear ex post equilibrium of the random supply auction than they are in the symmetric large supply equilibrium of the private information auction. Random supply bids are in dash-dotted red, private information bids are in dashed yellow, $n = 5$ bidders.

Corollary 4 directly implies that large supply equilibrium bids satisfy equation (7) and are equilibria in the models of Klemperer and Meyer (1989) and Ausubel et al. (2014). For further comparison we restrict attention to the "linear demand" case analyzed by these papers. When marginal values are given by $v(q) = 1 - q$ the equivalent asymptotic demand is a uniform distribution, $F(x) = x$.

Proposition 8 *(Comparison of divisible-good equilibria). In the linear ex post equilibrium of the random supply auction with linear demand, bids are given by* b^{RS}*,*

$$b^{RS}(q) = 1 - \left(\frac{n-1}{n-2}\right) q.$$

In the large supply equilibrium of the private information auction with linear demand, bids are given by b^{PI}*,*

$$b^{PI}(q) = \frac{1}{2}(1-q).$$

In a symmetric equilibrium of an auction satisfying market balance, $q_i^\star = (n-1)/n$. Then equilibrium prices are[41]

$$p^{\star,\mathrm{RS}} = -\frac{1}{n^2 - 2n}, \quad p^{\star,\mathrm{PI}} = \frac{1}{2n}.$$

Crucially, the market clearing price in the random supply auction is below the market clearing price in the (large supply) private information auction. Per equation (7) this implies that bids in the random supply auction must be steeper than bids in the private information auction. This is illustrated in Fig. 6.

[41] Because equilibrium in both auctions is sustained under additive translations of values the negative price in the random supply auction is not of concern: it can be made positive by shifting all values upward. Performing this translation within our model would require that the value distribution be supported strictly above zero, so we retain the negative price for expositional consistency.

In contrast with the unique separating equilibrium we obtain in the private information auction, the linear random supply equilibrium is unique in the class of linear equilibria; the uniform price auction with random supply may admit nonlinear equilibria. However, in all equilibria of the random supply auction equilibrium bids are initially equal to marginal values (i.e., $b(0) = v(0)$).[42] This clearly distinguishes the random supply auction from the private information auction, in which initial bids must be strictly below values.

The two auction models differ in crucial ways, most obviously that the pricing rules are distinct. A further difference addresses the question of who bidders are competing against. Consider the case of $n = 2$ bidders. With private information and known supply, a bid for unit $k_i = 1$ is competing against the opponent's bid for unit $k_{-i} = m$. The distribution of values implies that there is little incentive to bid aggressively for the first unit. With full information and random supply, a bid for unit $k_i = 1$ is competing against the opponent's bid for unit $k_{-i} = 1$. The distribution of values and the pricing rule imply standard Bertrand competition, and bids are initially equal to values. In light of this argument, conventional wisdom that uniform-price auctions have relatively steep bids may be amenable to further analysis.

5. Conclusion

We have defined a model of multi-unit auctions in which bidders have private values given by ordered draws from a single distribution. In this model, we show that the last accepted bid uniform-pricing rule induces bidding incentives analogous to those in a single-unit first price auction. We show that the last accepted bid auction can admit a tractable representation and can be both efficient and fully-revealing of bidders' private information. By noting the connection between bidding incentives in a single-unit first price auction and a multi-unit last accepted bid auction, we identify a new salient feature common to both auctions: in both auctions, bidders pay the highest market clearing price.

We compare the last accepted bid auction to the first rejected bid uniform price and the pay as bid auctions. We show that in each of these auctions bidder information is confounded in natural classes of equilibria. This further implies that these auctions are generally inefficient. We provide an additional construction which emphasizes that the first rejected bid auction always admits low-revenue equilibria, a phenomenon which cannot be sustained in the last accepted bid auction.

Although it may be true that for a fixed set of strategies the bidders' payoffs are not affected much by choosing the last accepted bid or the first rejected bid as the clearing price, our results show that the choice of clearing price does have a significant effect on equilibrium strategies. The underlying reason is that best response bids are determined by conditioning on the low probability event that particular bid is selected as the clearing price, and focusing on this event, our analysis makes clear that whether the clearing price is the last accepted bid or the first rejected bid has significant implications for how the bid is optimally chosen. Roughly, in a first rejected bid auction a bid is payoff-relevant only if it is submitted for a unit which *is not* won, while in a last accepted bid auction a bid is payoff-relevant only if it is submitted for a unit which *is* won; the differing alignments of bidding incentives lead to drastically different bidding behavior. Since

[42] Typically equilibrium will be nonlinear whenever the bid for the maximum-possible quantity is above the bid given in Klemperer and Meyer (1989) and Ausubel et al. (2014). In these models nonlinearity hinges on the full support of the random quantity realization (Woodward, 2019). Our linear equilibrium, with a higher maximum bid, does not contradict these results, since supply is fixed.

our model accommodates any number of units, the equilibrium in our model further provides a natural equilibrium selection in the divisible-good case, where the equilibrium need not be unique.

In symmetric single-unit auctions, efficiency and revenue typically move in opposite directions: when behavior is symmetric awarding the object to the highest bidder is efficient, so revenue can only be improved by conditionally not allocating the object (as with a reserve price). Thus revenue can increase as efficiency falls through nonallocation. In multi-unit auctions this connection breaks down. As we show in Section 4 inefficiency can arise by misallocation of goods. Then even in a symmetric equilibrium it is possible for revenue and efficiency to be simultaneously improved. We do not address this in this paper, but we believe the potential revenue gains of the last accepted bid auction over other multi-unit auction formats are worthy of future study.[43]

Taken as a whole, our results are strongly in favor of employing the last accepted bid pricing rule rather than the first rejected bid pricing rule when a uniform price auction is implemented. To date the literature has overlooked the possibility of a meaningful difference between the two; we show that this difference is real and has material implications in support of the last accepted bid auction.

Acknowledgments

We gratefully acknowledge discussions and feedback from Larry Ausubel, Peter Cramton, Paul Klemperer, Pino Lopomo, Meg Meyer, David McAdams, Marek Pycia, and Curt Taylor, seminar audiences at INFORMS, Duke IO Theory, SEA, Collegio Carlo Alberto, and Oxford, and several anonymous referees.

Appendix A. First-order conditions for leading example

A.1. Last accepted bid

Each bidder's utility can be expressed as

$$\begin{aligned}u^i\left(b^i;v^i\right) &= v_1^i F_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) + v_2^i F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right)\\ &\quad - \left(F_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) - F_{(1)}\left(\varphi_1^{-i}\left(b_1^i\right)\right)\right) b_1^i - 2b_2^i F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right)\\ &\quad - \int\limits_{\varphi_1^{-i}\left(b_2^i\right)}^{\varphi_1^{-i}\left(b_1^i\right)} b_1^{-i}(v)\, dF_{(1)}(v).\end{aligned}$$

From here, it is straightforward to compute the model's first order conditions,

$$\frac{\partial}{\partial b_1^i} : \left(v_1^i - b_1^i\right) dF_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) d\varphi_2^{-i}\left(b_1^i\right) - \left(F_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) - F_{(1)}\left(\varphi_1^{-i}\left(b_1^i\right)\right)\right);$$

[43] In Section 4 we showed that zero revenue occurs with positive probability in all equilibria of the first rejected bid auction, which does not establish a ranking in expected revenue. In Section 1.1 we show that all equilibria of the particular first rejected bid auction have zero expected revenue, which does not establish a ranking in general models.

$$\frac{\partial}{\partial b_2^i} : \left(v_2^i - b_2^i\right) dF_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right) d\varphi_1^{-i}\left(b_2^i\right) - 2F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right).$$

Assuming a symmetric equilibrium many bid decorators can be dropped; substituting in for the known order statistic distributions ($F_{(1)}(x) = x^2$, $F_{(2)}(x) = 2x - x^2$) gives

$$2\left(\varphi_1(b) - b\right)\left(1 - \varphi_2(b)\right) d\varphi_2(b) - \left(2\varphi_2(b) - \varphi_2(b)^2 - \varphi_1(b)^2\right) = 0;$$

$$\left(\varphi_2(b) - b\right)\varphi_1(b)\, d\varphi_1(b) - \varphi_1(b)^2 = 0.$$

A.2. First rejected bid

Each bidder's utility can be expressed as

$$\begin{aligned} u^i\left(b^i; v^i\right) = {} & v_2^i F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right) - 2\int_0^{\varphi_1^{-i}(b_2^i)} b_1^{-i}(v)\, dF_{(1)}(v) \\ & - \left(F_{(2)}\left(\varphi_2^{-i}\left(b_2^i\right)\right) - F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right)\right) b_2^i \\ & + v_1 F_{(2)}\left(\varphi_2^{-i}(b_1)\right) - \int_{\varphi_2^{-i}(b_2^i)}^{\varphi_2^{-i}(b_1^i)} b_2^{-i}(v)\, dF_{(2)}(v). \end{aligned}$$

From here, it is straightforward to compute the model's first-order conditions,

$$\frac{\partial}{\partial b_1^i} : \left(v_1^i - b_1^i\right) dF_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) d\varphi_2^{-i}\left(b_1^i\right);$$

$$\frac{\partial}{\partial b_2^i} : \left(v_2^i - b_2^i\right) dF_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right) d\varphi_1^{-i}\left(b_2^i\right) - \left(F_{(2)}\left(\varphi_2^{-i}\left(b_2^i\right)\right) - F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right)\right).$$

The first order condition with respect to the bid for the first unit, b_1^i, confirms the intuition that truthful reporting is a weakly dominant strategy. This follows from standard second price auction logic: the bid for the first unit never sets the clearing price (when the agent wins) so it is effectively costless to increase the bid.

In an equilibrium in which agents bid truthfully for their initial units the first order condition with respect to the second-unit bid is no longer (meaningfully) a differential equation: $\varphi_1(b) = b$, and hence $d\varphi_1(b) = 1$. Substituting through, symmetric equilibrium bids for the second unit must solve

$$2\left(v_2 - b\right) b - \left(2v_2 - v_2^2 - b^2\right) = 0, \quad \text{or} \quad b = v_2 \pm \sqrt{2v_2^2 - 2v_2}. \tag{8}$$

Since $v_2 \in [0, 1]$, it must be that $2v_2^2 - 2v_2 \leq 0$; then the negative quadratic in (8) has no real zeroes, and the first order condition with respect to b_2 is negative everywhere. It follows that $b_2 = 0$ identically, independent of v_2.

Appendix B. Proofs of equilibrium properties

Proof of Lemma 4. This is implied by continuity and differentiability of the bid distribution functions. These results are similar to the arguments familiar from the first price auction, but we reproduce them here due to the changes in the bidders' utility functions induced by shifting to a multi-unit model. We say that unit k is *opposed to* unit $m-k+1$, in the sense that agent i wins unit k if and only if agent $j \neq i$ wins unit $m-k+1$. Recall the separable utility representation for the LAB auction,[44]

$$u^i(b; v) = \sum_{k=1}^{m} v_k H_{m-k+1}^{-i}(b_k) - \left(H_{m-k+1}^{-i}(b_k) - H_{m-k}^{-i}(b_k)\right) k b_k \\ - k \int_0^{b_k} x dH_{m-k}^{-i}(x) + (k-1) \int_0^{b_k} x dH_{m-k+1}^{-i}(x).$$

First, there are no gaps in equilibrium bid distribution functions. If there is a gap in H_{m-k+1}^{-i}, then a bid for agent i's unit k strictly inside this gap induces no additional winning probability but incurs additional expected costs (vis-à-vis bidding just above the lower bound). It follows that any gaps in H_{m-k+1}^{-i} are shared by the opposing distribution H_k^i. Since there is no probability gain within the gap, for a bid to be placed at the upper end of the gap there must be a mass point[45]; there are therefore identical mass points for the opposing units k and $m-k+1$. Identical mass points cannot arise for standard tiebreaking reasons, therefore this is not supportable in equilibrium.

Second, there are no mass points in equilibrium bid distributions (i.e., equilibrium bid distributions are continuous). Suppose that there is a mass point in H_{m-k+1}^{-i} at bid b, but no mass point in H_{m-k}^{-i}. Since bids are in general strictly below values[46] and there are no gaps in the bid distributions, there is a value v such that $b_k^i(v) = b - \varepsilon$ for any $\varepsilon > 0$. For ε small enough, a slight increase to $\tilde{b}_k^i(v) = b + \varepsilon$ yields a discrete jump in expected utility; this implies that gaps exist in response to mass points, and we have already established that gaps cannot exist. Otherwise, suppose that there are mass points in both H_{m-k+1}^{-i} and H_{m-k}^{-i} at b, so that the above logic does not apply. However, if this is the case, then there is a mass point in H_{m-k}^{-i}, the unit opposed to bidder i's unit $k+1$. Then the previous argument holds unless there is also a mass point in H_{m-k-1}^{-i}, and so on. Since there are no mass points in the degenerate distribution H_0^{-i} — the $H_{m-\tilde{k}}^{-i}$ corresponding to $\tilde{k} = m$ — the original argument must hold for some unit, violating the no-gaps property established above.

Since equilibrium bid distributions are continuous and differentiable, the first order conditions must be satisfied in any equilibrium in separating strategies. $\square$

44 This expression appears to presuppose the differentiability of $H_{k'}^{-i}$ for all k', however it is a re-expression of one in terms of well-defined conditional expectations; since we establish that in equilibrium the bid distributions are continuously differentiable this expression is ultimately correct. We do not presuppose the correctness of this expression, and avoid this potential circularity in our formal arguments.

45 This analysis ignores the possibility that the support of the bid distribution above the gap is left-open. For a bid sufficiently close to this upper endpoint, the arguments are the same.

46 This somewhat obvious point is proved explicitly in an earlier version of this paper, and is familiar from results in single-unit auctions.

Proof of Proposition 11. When bids are separating there can be no mass points in the bid distribution. This proof proceeds by ruling out gaps in first-unit bids, then successively ruling out differently-oriented kinks in bids.[47]

Recall from Appendix C that separable payoffs for bids for the two units are

$$u^1(b;v) = (v_1 - b_1) H_2(b_1) + \int_0^{b_1} H_1(x)\,dx,$$

$$u^2(b;v) = (v_2 - b_2) H_1(b_2) - \int_0^{b_2} H_1(x)\,dx.$$

The relevant probabilities are

$$H_1(x) = F_{(1)}(\varphi_1(x))^{n-1},$$
$$H_2(x) = (n-1) F_{(1)}(\varphi_1(x))^{n-2} F_{(2)}(\varphi_2(x)) \left(1 - F_{(1)}(\varphi_1(x))\right) + H_1(x).$$

Suppose that there is a gap (discontinuity) in a symmetric equilibrium first-unit bid function b_1. As is clear from the definition of unitwise utility, second-unit bids will never be placed in this gap; H_1 is constant on this gap so unitwise utility is strictly decreasing. Then if there is a gap in the first-unit bid there is a gap in the aggregate market clearing price distribution. As in other auction models, there is no incentive to bid just above the upper bound of the common gap, implying that this is not possible in an equilibrium without mass points.

Now suppose that there is a downward kink in b_1 at q, so that

$$\lim_{\varepsilon \searrow 0} \frac{b_1(q) - b_1(q-\varepsilon)}{\varepsilon} > \lim_{\varepsilon \searrow 0} \frac{b_1(q+\varepsilon) - b_1(q)}{\varepsilon}.$$

Since second-unit bids depend only on the first-unit bid function and there are no mass points, there must be a gap above q in the second-unit bid b_2. Then $F_{(2)} \circ \varphi_2$ is constant just above q, and first-unit bids depend only on $F_{(1)} \circ \varphi_1$. The downward kink in b_1 then implies either a mass point or a gap in b_1 just above q, which we have shown cannot arise.

Now suppose there is an upward kink in b_1 at q, so that

$$\lim_{\varepsilon \searrow 0} \frac{b_1(q) - b_1(q-\varepsilon)}{\varepsilon} < \lim_{\varepsilon \searrow 0} \frac{b_1(q+\varepsilon) - b_1(q)}{\varepsilon}.$$

Since the monotonicity constraint is not binding, this implies a gap in b_2 just above q. Per the previous argument regarding the downward kink, this is not possible in equilibrium.

Since there are no kinks in b_1 it is differentiable; this implies (via the same arguments as above) that b_2 is differentiable — nondifferentiabilities in b_2 will manifest in first-unit incentives, inducing gaps or mass points, which cannot arise. Then b_1 and b_2 are differentiable on their support, and the first order conditions must be satisfied in a symmetric separating equilibrium. □

[47] In a monotone separating equilibrium, the quantity-monotonicity constraint will never strictly bind. This proof can be adapted to allow for binding monotonicity constraints, but it is not necessary to the point at hand.

Appendix C. Proofs of information pooling properties

Lemma 9 *(Constrained bids strictly between optimal unconstrained bids). Suppose that bidder i with type v^i submits a constant bid $b^i_{\{k,\dots,k+a\}}$ for units $k,\dots,k+a$ and let $x^i_l(v^i_l)$ and $x^i_s(v^i_s)$ with $l,s\in\{k,\dots,k+a\}$ be respectively any of the bidder's largest and smallest unconstrained bids for these units. Then $x^i_l(v^i_l) > x^i_s(v^i_s)$ implies $x^i_l(v^i_l) > b^i_{\{k,\dots,k+a\}} > x^i_s(v^i_s)$.*

Proof. The first order condition for the constrained bid is

$$\sum_{y=k}^{k+a} \frac{\partial}{\partial b^i_y} u^i(b^i_{\{k,\dots,k+a\}}; v^i) = 0, \tag{9}$$

or the sum of the unconstrained bid first order conditions. Note that the objective is quasi-concave in each b^i_y. At the largest unconstrained bid, x^i_l, the first-order conditions for the other bids cannot be positive, due to quasi-concavity, and given $x^i_l(v^i_l) > x^i_s(v^i_s)$ at least one is negative. Therefore, at x^i_l, the left-hand side of (9) is negative. A similar argument implies that the left-hand side of (9) is positive at x^i_s. □

Proof of Lemma 2. We analyze each auction in turn. Note that it is without loss in each case to consider the agent as bidding for all m available units, with the constraint that she has zero value for units $k > m_i$.

FRB. Utility is written as

$$\begin{aligned} u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} & \left(\sum_{k'=1}^{k} v_{k'} - k b^i_{k+1}\right) \Pr\left(b^{-i}_{m-k} \ge b^i_{k+1} \ge b^{-i}_{m-k+1}\right) \\ & + \left(\sum_{k'=1}^{k} v_{k'} - k\mathbb{E}\left[b^{-i}_{m-k+1} \middle| b^i_k \ge b^{-i}_{m-k+1} \ge b^i_{k+1}\right]\right) \\ & \times \Pr\left(b^i_k \ge b^{-i}_{m-k+1} \ge b^i_{k+1}\right). \end{aligned}$$

The relevant probabilities are

$$\begin{aligned} \Pr\left(b^{-i}_{m-k} \ge b^i_{k+1} \ge b^{-i}_{m-k+1}\right) &= H^{-i}_{m-k+1}\left(b^i_{k+1}\right) - H^{-i}_{m-k}\left(b^i_{k+1}\right), \\ \Pr\left(b^i_k \ge b^{-i}_{m-k+1} \ge b^i_{k+1}\right) &= H^{-i}_{m-k+1}\left(b^i_k\right) - H^{-i}_{m-k+1}\left(b^i_{k+1}\right). \end{aligned}$$

Algebraic manipulation gives a separable utility form of[48]

$$u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} v_k H^{-i}_{m-k+1}\left(b^i_k\right) - \left(H^{-i}_{m-k+2}\left(b^i_k\right) - H^{-i}_{m-k+1}\left(b^i_k\right)\right)(k-1)\, b^i_k$$

---

[48] This form is a convenient symmetric shorthand, but H^{-i}_{m+1} is ill-defined. Since this term is in $[0, 1]$ and is always premultiplied by $(1-1)=0$, the exact specification is irrelevant.

$$- k \int_0^{b_k^i} b d H_{m-k+1}^{-i} + (k-1) \int_0^{b_k^i} b d H_{m-k+2}^{-i}.$$

LAB. Utility is written as

$$u^i \left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} \left(\sum_{k'=1}^{k} v_{k'} - k b_k^i \right) \Pr\left(b_{m-k}^{-i} \geq b_k^i \geq b_{m-k+1}^{-i}\right)$$
$$+ \left(\sum_{k'=1}^{k} v_{k'} - k \mathbb{E}\left[b_{m-k}^{-i} \middle| b_k^i \geq b_{m-k}^{-i} \geq b_{k+1}^i \right] \right)$$
$$\times \Pr\left(b_k^i \geq b_{m-k}^{-i} \geq b_{k+1}^i\right).$$

The relevant probabilities are

$$\Pr\left(b_{m-k}^{-i} \geq b_k^i \geq b_{m-k+1}^{-i}\right) = H_{m-k+1}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_k^i\right),$$
$$\Pr\left(b_k^i \geq b_{m-k}^{-i} \geq b_{k+1}^i\right) = H_{m-k}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_{k+1}^i\right).$$

Algebraic manipulation gives a separable utility form of

$$u^i \left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} v_k H_{m-k+1}^{-i}\left(b_k^i\right) - \left(H_{m-k+1}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_k^i\right)\right) k b_k^i$$
$$- k \int_0^{b_k^i} b d H_{m-k}^{-i} + (k-1) \int_0^{b_k^i} b d H_{m-k+1}^{-i}.$$

PAB. This is essentially trivial. Note that utility has a naturally separable form,

$$u^i \left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m_i} \left(v_k - b_k^i\right) H_{m-k+1}^{-i}\left(b_k^i\right). \quad \square$$

Proof of Lemma 5. Note that strict separation implies that strategies are strictly monotone in value. If strategies cannot be separated as in the statement of the Lemma, the monotonicity constraint must be binding.[49]

Suppose first that bids are continuous in value. If the bid profile cannot be written as a product of independent dimensional bids, the monotonicity constraint must be binding over some product of nondegenerate intervals $[\underline{b}_k, \overline{b}_k) \times \cdots \times [\underline{b}_{k'}, \overline{b}_{k'})$. Because bids are continuous in value, bids are not invertible on this range; since this range has positive measure (and can be expanded to account for higher and lower units for which the monotonicity constraints are not binding) it follows that equilibrium is not strictly separating, a contradiction.

[49] Due to standard peculiarities of measure zero, this statement is true when constrained to left-continuous bid functions but may fail in the presence of arbitrary discontinuities. It is straightforward to show that any monotone strictly separating equilibrium is incentive-equivalent to an equilibrium which is left-continuous in value, thus this statement is more or less without loss of generality.

Now suppose that the monotonicity constraint is binding at a point at which bids are discontinuous in value. Because dimensional utilities satisfy increasing differences and are continuous in value, there is a neighborhood below this point on which the monotonicity constraint is binding and bids are locally continuous. Then the above argument holds. □

Proof of Lemma 6. The unitwise utility function in the FRB auction can be expressed as

$$u_k^i\left(b_k^i, b^{-i}; v_k\right) = \left(v_k - b_k^i\right) H_{m-k+1}^{-i}\left(b_k^i\right) - (k-1)\int_0^{b_k^i} H_{m-k+2}^{-i}(x) - H_{m-k+1}^{-i}(x)\,dx.$$

Since for any unit the agent has the option of bidding 0 and obtaining (at worst) zero utility, $u_k^i(b_k^i(v), b^{-i}; v_k) \geq 0$ whenever b_k^i is a best response bidding function. As established in Lemma 5, if equilibrium does not exhibit partial pooling, $b_k^i(v) \equiv b_k^i(v_k)$. It follows that in an equilibrium without partial pooling,

$$\frac{v_k - b_k^i(v_k)}{k-1} \geq \frac{\int_0^{b_k^i(v_k)} H_{m-k+2}^{-i}(x) - H_{m-k+1}^{-i}(x)\,dx}{H_{m-k+1}^{-i}\left(b_k^i(v_k)\right)}.$$

Strict separation, well-behavedness, and the assumption that b^i is a best response require that $b_k^i(v_k) > 0$ whenever $v_k > 0$, that bids are dense near 0, and that $b_k^i(0) = 0$.[50] Then in the limit, for all $k > 1$,[51]

$$\lim_{b \searrow 0} \frac{\int_0^b H_{m-k+2}^{-i}(x) - H_{m-k+1}^{-i}(x)\,dx}{H_{m-k+1}^{-i}(b)} = 0.$$

When equilibrium is well-behaved and arbitrarily differentiable, for a set of relevant $t \in \{0, 1, \ldots, \bar{t}\}$ l'Hôpital's rule implies[52]

$$\lim_{b \searrow 0} \frac{d^{(t)} H_{m-k+2}^{-i}(b) - d^{(t)} H_{m-k+1}^{-i}(b)}{d^{(t+1)} H_{m-k+1}^{-i}(b)} = 0.$$

Bid monotonicity requires that $b_{k'}^j(v) > b_{k'+1}^j(v)$, and hence by the nature of the order statistic model there is some t such that

$$\lim_{b \searrow 0} \left| d^{(t)} H_{m-k+2}^{-i}(b) \right| > \lim_{b \searrow 0} \left| d^{(t)} H_{m-k+1}^{-i}(b) \right| = 0.$$

At this t, well-behavedness requires that $\lim_{b \searrow 0} |d^{(t+1)} H_{m-k+1}^{-i}(b)| \geq 0$ is finite, hence the limit is strictly positive, contradicting strict separation. □

50 In a working version of this paper we provide arguments that these statements continue to hold in the presence of a reserve price $r > 0$.

51 Technically only a weak inequality, ≤ 0, is required. Given the relationship between H_{m-k+2}^{-i} and H_{m-k+1}^{-i} it is straightforward to show that strict inequality cannot be satisfied.

52 This limit makes clear the hidden role of the assumption that $m \geq 2$ units are available. When $m = 1$, $H_{m-k+2}^{-i} = 0$, invalidating this proof approach. This is to be expected, since with $m = 1$ unit available the FRB auction is equivalent to a second price auction, which admits a well-behaved, separating, truthful equilibrium.

Proof of Lemma 7. Let $\bar{b}_k^i$ be bidder i's maximum bid for unit k; without loss of generality, this is $\bar{b}_k^i = b_k^i(\bar{v})$. Suppose that $\bar{b}_k^i \neq \bar{b}_{m-k+1}^{-i}$, and without loss of generality assume that $\bar{b}_k^i > \bar{b}_{m-k+1}^{-i}$. Then anytime bidder i submits a bid $b \in (\bar{b}_{m-k+1}^{-i}, \bar{b}_k^i]$, she wins unit k with probability 1; she could reduce her bid without affecting her winning probability, improving her utility. Then $\bar{b}_k^i = \bar{b}_{m-k+1}^{-i}$ for all units k.

Bid monotonicity requires that $\bar{b}_k^i \geq \bar{b}_{k'}^i$ for all $k' \geq k$. Then

$$\bar{b}_k^i \geq \bar{b}_{k'}^i = \bar{b}_{m-k'+1}^{-i} \geq \bar{b}_{m-k+1}^{-i} = \bar{b}_k^i.$$

Then $\bar{b}_k^i = \bar{b}_{k'}^i$ for all $k' \geq k$, and bidder i's maximum bid is independent of the unit she is bidding for. Since this maximum bid is equal to bidders' $-i$ maximum bid for the complementary unit, the maximum bid is independent of unit and agent. □

Proof of Lemma 8. If bidder i's best-response bid function for unit k, $b_k^\star$, is continuous, it must be that H_{m-k+1}^{-i} is continuous; moreover, where H_{m-k+1}^{-i} is not differentiable it has a "downward" kink. Following the first order conditions of the model, for any $v \in (0, 1)$ it must be that either[53]

$$b_k^\star(v) > b_{k+1}^\star(v), \text{ or } d_+ H_{m-k}^{-i}\left(b_k^\star(v)\right) < d_+ H_{m-k+1}^{-i}\left(b_k^\star(v)\right) \text{ (or both).}$$

Since $b_{\tilde{k}}^\star$ is continuous for each $\tilde{k}$ (by assumption), whenever the first inequality holds it must hold over an interval. Note that it must be that there is some v for which the first inequality holds; otherwise $b_k^\star(v) = b_{k+1}^\star(v)$ for all v, implying the second inequality for all $b \in [0, \bar{b}]$. Then there is a mass point in H_{m-k}^{-i} at $\bar{b}$, which is inconsistent with i submitting a continuous bid function.[54]

Let v be such that $b_k^\star(v) > b_{k+1}^\star(v)$. Appealing to incentive compatibility and first-order dominance,

$$\begin{aligned}\left(v - b_k^\star(v)\right) H_{m-k+1}^{-i}\left(b_k^\star(v)\right) &\geq \left(v - b_{k+1}^\star(v)\right) H_{m-k+1}^{-i}\left(b_{k+1}^\star(v)\right)\\ &> \left(v - b_{k+1}^\star(v)\right) H_{m-k}^{-i}\left(b_{k+1}^\star(v)\right)\\ &\geq \left(v - b_k^\star(v)\right) H_{m-k}^{-i}\left(b_k^\star(v)\right).\end{aligned}$$

These inequalities imply

$$\frac{H_{m-k+1}^{-i}\left(b_k^\star(v)\right)}{H_{m-k}^{-i}\left(b_k^\star(v)\right)} > \frac{H_{m-k+1}^{-i}\left(b_{k+1}^\star(v)\right)}{H_{m-k}^{-i}\left(b_{k+1}^\star(v)\right)}.$$

Continuity and maximality imply that there is $\tilde{v}$ with $b_{k+1}^\star(\tilde{v}) = b_k^\star(v)$. Then

$$\frac{H_{m-k+1}^{-i}\left(b_k^\star(\tilde{v})\right)}{H_{m-k}^{-i}\left(b_k^\star(\tilde{v})\right)} \geq \frac{H_{m-k+1}^{-i}\left(b_{k+1}^\star(\tilde{v})\right)}{H_{m-k}^{-i}\left(b_{k+1}^\star(\tilde{v})\right)}.$$

[53] The use of $b_k^\star$ (instead of $b_{k+1}^\star$) is irrelevant here, and is used solely to ensure that attention is focused on a single bid. It is also sufficient to consider only right derivatives, which are finite at all relevant points (otherwise a slight increase in bid would trivially be profitable, and would be feasible since bids are necessarily below values whenever right derivatives are nonzero).

[54] Either i's bid function is discontinuous, or the high bid is such that $\bar{b} = 1$. In this latter case, the existence of a mass point implies some of i's opponents are bidding above their values and winning with positive probability, which is not a best response in the PAB auction.

Let $I(b)$ be the interval over which $b_k^\star(v') > b_{k+1}^\star(v')$,

$$I(b) = \begin{bmatrix} \inf\left\{b_k^\star(v') : b_k^\star(\tilde{v}') > b_{k+1}^\star(\tilde{v}')\, \forall \tilde{v}' \in (v', v]\right\}, \\ \sup\left\{b_k^\star(v') : b_k^\star(\tilde{v}') > b_{k+1}^\star(\tilde{v}')\, \forall \tilde{v}' \in [v, v')\right\} \end{bmatrix}.$$

The preceding inequalities and standard sequential arguments imply that the difference between the CDFs H_{m-k+1}^{-i} and H_{m-k}^{-i} is maximized at the interval's right endpoint,

$$\begin{aligned} &H_{m-k+1}^{-i}\left(\min I\left(b_k^\star(v)\right)\right) - H_{m-k}^{-i}\left(\min I\left(b_k^\star(v)\right)\right) \\ &\quad < H_{m-k+1}^{-i}\left(\max I\left(b_k^\star(v)\right)\right) - H_{m-k}^{-i}\left(\max I\left(b_k^\star(v)\right)\right). \end{aligned}$$

The right endpoint of the interval is either the left endpoint of another interval on which $b_k^\star(\tilde{v}) > b_{k+1}^\star(\tilde{v})$, or of an interval on which $d_+ H_{m-k}^{-i}(b) < d_+ H_{m-k+1}^{-i}(b)$. In the latter case, the difference between the two CDFs is again maximized at the right endpoint of the subsequent interval (and the former case is as analyzed above). In either case, the difference between the CDFs at the right endpoint is increasing in the location of the interval. Since $H_{m-k+1}^{-i} \succeq_{\text{FOSD}} H_{m-k}^{-i}$, it follows that $H_{m-k}^{-i}(\bar{b}) < H_{m-k+1}^{-i}(\bar{b})$ and hence H_{m-k}^{-i} has a mass point at $\bar{b}$. □

Proof of Corollary 3. This is a direct consequence of Lemmas 7 and 8. At $\bar{b}$, $H_{m-k+1}^{-i}(\bar{b}) > H_{m-k}^{-i}(\bar{b})$, contradicting best response behavior. □

Appendix D. *n* bidders and two units

If the market is either balanced and bidders have symmetric demands or there are two bidders with asymmetric demand for all units, we can identify equilibrium strategies with a corresponding first price auction. This is no longer true in the general case where the market is either unbalanced or there are more than two asymmetric bidders. A common property of the equilibria analyzed in the main text is that there exists a univariate function which bidder i uses to determine the bids on all of her marginal units from their marginal values. In general this property, which allows for the reduction to a first price auction, does not hold in equilibrium, and bidders may shade their bids on marginal units differently depending on the unit to which the bid is on.

Despite not being able to explicitly characterize equilibrium strategies in the general case, we show in this appendix that we can utilize techniques from the first price auctions literature to establish that some key properties still hold. For example, we present an unbalanced, symmetric demand case in which we can prove a uniqueness result for equilibrium strategies using an argument that closely resembles the uniqueness argument typically given for the equilibrium of a first-price auction. Existing literature ensures the existence of a pure-strategy equilibrium.

Proposition 9 *(Equilibrium existence in LAB). With $n \geq 2$ bidders $i \in \{1, \ldots, n\}$, where bidder i's m_i marginal values are the order statistics from the distribution F^i, the LAB auction admits a pure strategy Bayesian Nash equilibrium.*

Proof. This follows from Corollary 5.2 in Reny (2011).[55] □

[55] Reny (2011) investigates the FRB auction. With regard to existence (although not, as argued above, the structure of equilibrium) the arguments do not change in a substantive way.

Suppose henceforth that there are n symmetric bidders, each with demand for $m_i = 2$ units, and there are $m = 2$ units available.

D.1. Uniqueness

We first evaluate the uniqueness of symmetric equilibria in this context. Since the case with $n = 2$ bidders is covered by the analysis in the main text we assume that $n \geq 3$. Let $b_1(v_1)$ and $b_2(v_2)$ represent a candidate equilibrium, where $b_1(v) \geq b_2(v)$ for all $v \in (0, 1)$. Denote the inverse bid functions by φ_1 and φ_2 respectively.

The argument closely resembles the uniqueness argument for the $n = 2$ bidder, balanced market case given in Section 3.1.2, which in turn resembles uniqueness arguments given for asymmetric first price auctions. A key difference that arises is that with more than two bidders it is no longer necessarily true that there is a common high bid for each marginal unit. One important implication of proving that there *is* a common high bid for each marginal unit is that the initial conditions for the system of differential equations derived from the first order conditions have a single degree of freedom.

We first show that the initial conditions in this case have a single degree of freedom as well, despite the fact that $b_1(1) > b_2(1)$ (see Corollary 5). When $b_1(1) > b_2(1)$, there are two distinct intervals of bids between which the first order conditions for the optimal bids change. Bids $b \in [0, b_2(1)]$ compete against first- and second-unit bids made by opponents. That values are given by ordered draws implies $F_{(1)}(v) \equiv F(v)^2$, $H_1(b) \equiv F_{(1)}(\varphi_1(b))$, $F_{(2)}(v) \equiv 2F(v) - F(v)^2$, and $H_2(b) \equiv F_{(2)}(\varphi_2(b))$. The first order conditions for the first- and second-unit bids in this range simplify to

$$\left(\frac{h_2(b)}{H_2(b) - H_1(b)} + \frac{(n-2)\,h_1(b)}{H_1(b)}\right)(v_1 - b) = 1, \text{ and} \tag{10}$$

$$\frac{(n-1)\,h_1(b)}{2H_1(b)}(v_2 - b) = 1. \tag{11}$$

For bids $b \in (b_2(1), b_1(1)]$, the opposing bids are solely for the first unit, changing the win probabilities for a bidder's first unit. The corresponding first order condition for the first unit is

$$\frac{(n-2)\,h_1(b)}{H_1(b)}(v_1 - b) = 1. \tag{12}$$

Since only first-unit bids are submitted in $(b_2(1), b_1(1)]$ and assuming that bidders use symmetric strategies, the bid function that solves (12) can be represented explicitly up to an unknown bid, because it reduces to the best response ODE of a symmetric first price auction with $n - 1$ total bidders.[56] If $b_1(1)$ is known, the unique solution to this ODE can be represented as

$$b_1(1) - b_1(v_1)F(v_1)^{n-2} = \int_{v_1}^{1} x\, dF(x)^{n-2}. \tag{13}$$

Equation (13) is an explicit characterization of first-unit bids on the interval $(b_2(1), b_1(1)]$ for a known value of $b_1(1)$. We next argue that the value of $b_2(1)$ is pinned down in equilibrium by $b_1(1)$. Note that the first order condition for the first unit, equation (12), is the same as the

[56] See our Definition 4.

one in equation (10) with $h_2(b) = 0$. We also observe that it must be that $h_2(b_2(1)) = 0$ because the density of the second order statistic vanishes at the upper bound of its support. The inverse bid function associated with the solution in equation (13) therefore satisfies equation (10) in a neighborhood of $b_2(1)$. This implies that a necessary condition for the selection of $b_2(1)$ is that it be optimally chosen according to equation (11) where $H_1(b)$ is the bid distribution determined by the initial choice of $b_1(1)$ and equation (13). In other words, at $b_2(1)$ the values of h_1 and H_1 are known up to $b_1(1)$.

In the uniqueness argument given in Section 3.1.2, the second step is to show that given two distinct initial conditions for the ODE derived from two distinct choices for the common high bid, the corresponding solutions to the ODE are monotonic in these initial conditions at all points in the interior of the domain. The same property holds in the $n \times 2$ case, which we record as Lemma 10. Similar to our earlier analysis, we view equations (10) and (11) as an ODE in unknown H_1 and H_2 with domain $(0, b_2(1)]$ and initial conditions determined by the value taken by $b_2(1)$ and the value of $\bar{v}_1 \equiv \varphi_1(b_2(1))$, where φ_1 is determined at $b_2(1)$ by equation (13). Using equations (10) and (11) and letting $\varphi_k \equiv F_k^{-1} \circ H_k$, this ODE can be expressed as

$$\frac{d}{db} H_2(b) = \frac{H_2(b) - H_1(b)}{\varphi_1(b) - b} - 2\left(\frac{n-2}{n-1}\right)\left(\frac{H_2(b) - H_1(b)}{\varphi_2(b) - b}\right), \text{ and} \tag{14}$$

$$\frac{d}{db} H_1(b) = \frac{1}{n-1}\left(\frac{2H_1(b)}{\varphi_2(b) - b}\right). \tag{15}$$

Equations (14) and (15) can be used to show that distinct solutions to the differential system never meet, except potentially at $v = 0$.

Lemma 10. *Let $\hat{b}_1(1) < \tilde{b}_1(1)$ be two initial choices for $b_1(1)$ and $\hat{b}_2(1) < \tilde{b}_2(1)$ be the corresponding choices for $b_2(1)$. Let $\hat{H}_1$ and $\hat{H}_2$ solve equations (14) and (15) when the initial condition is $\hat{H}_2(\hat{b}_2(1)) = 1$ and $\hat{H}_1(\hat{b}_2(1)) = F_{(1)}(\hat{v}_1)$, and let $\tilde{H}_1$ and $\tilde{H}_2$ solve equations (14) and (15) when the initial condition is $\tilde{H}_2(\tilde{b}_2(1)) = 1$ and $\tilde{H}_1(\tilde{b}_2(1)) = F_{(1)}(\tilde{v}_1)$. For all $b \in (0, \hat{b}_2(1))$, $\hat{H}_1(b) > \tilde{H}_1(b)$ and $\hat{H}_2(b) > \tilde{H}_2(b)$.*

Proof. Since the equilibrium bid functions are increasing we have $\hat{H}_k(\hat{b}_2(1)) = 1 > \tilde{H}_2(\hat{b}_2(1))$ for $k = 1, 2$. We show next that this inequality holds for all bids $b \in (0, \hat{b}_2(1)]$. To do this, we rule out that $\hat{H}_k$ crosses $\tilde{H}_k$ for either k at any point in the domain. Let $\hat{b} < b_2(1)$ represent the largest bid at which either $\hat{H}_1$ crosses $\tilde{H}_1$ or $\hat{H}_2$ crosses $\tilde{H}_2$. Consider first the case where $\hat{H}_1(\hat{b}) = \tilde{H}_1(\tilde{b})$ and $\hat{H}_2(\hat{b}) > \tilde{H}_2(\hat{b})$. Using equation (15), this implies that $\hat{h}_1(\hat{b}) < \tilde{h}_1(\hat{b})$, but this implies that $\tilde{H}_1$ crosses $\hat{H}_1$ from above, a contradiction. Similarly, one can show using equation (14) that $\hat{H}_1(\hat{b}) > \tilde{H}_1(\hat{b})$ and $\hat{H}_2(\hat{b}) = \tilde{H}_2(\hat{b})$ imply $\hat{h}_2(\hat{b}) < \tilde{h}_2(\hat{b})$. If it were true that $\hat{H}_1(\hat{b}) = \tilde{H}_1(\hat{b})$ and $\hat{H}_2(\hat{b}) = \tilde{H}_2(\hat{b})$ (i.e., they both crossed together), then we would have to conclude by the FTODE that $\hat{b}_2(1) = \tilde{b}_2(1)$, because the FTODE implies that there is a unique solution to the system in equations (14) and (15) beginning from an initial value at a $\hat{b} \in (0, \hat{b}_2(1)]$. □

Lemma 10 implies that any two solutions to equations (14) and (15) are ordered pointwise in the interior of the domain according to the ordering of the high bids on the first unit. The final step in the uniqueness proof is to use this fact to rule out that there can be more than one valid choice of $b_1(1)$.

In the literature on uniqueness in first price auctions, an additional assumption is required to complete this final step. This may be an assumption that l'Hôpital's rule can be applied to the

ODE at the low bid[57]; an assumption that there is a binding reserve price or an atom at the lower end of the support of values (Lebrun, 1999; Maskin and Riley, 2003); or an assumption about the properties of the value distribution in an interval including the lower bound of the support (Lebrun, 2006). Each of these approaches apply here as well, using the implications of Lemma 10 and the equation in (15).

Equation (15) implies that for two bids, $b < b'$,

$$\frac{H_1\left(b'\right)}{H_1\left(b\right)} = \exp\left\{\frac{2}{n-1}\int_b^{b'}\frac{dx}{\varphi_2\left(x\right)-x}\right\}. \tag{16}$$

As in Lemma 10, let $\hat{b}_2(1) < \tilde{b}_2(1)$ so that $\hat{H}_1(b) > \tilde{H}_1(b)$ and $\hat{H}_2(b) > \tilde{H}_2(b)$ for all $b \in (0, \hat{b}_2(1))$. From equations (16) and Lemma 10, it follows that

$$1 < \frac{\hat{H}_1\left(b'\right)}{\tilde{H}_1\left(b'\right)} < \frac{\hat{H}_1\left(b\right)}{\tilde{H}_1\left(b\right)}. \tag{17}$$

With an atom at the bottom of the distribution F equal to c it must be in equilibrium that $H_1(0) = c^2$, implying that if $\hat{H}_1$ and $\tilde{H}_1$ both derive from equilibrium strategies $\hat{H}_1(0)/\tilde{H}_1(0) = 1$. But this requirement conflicts with equation (17), which bounds this ratio away from one for all $b < b'$. We conclude that $\hat{H}_1$ and $\tilde{H}_1$ cannot both derive from equilibrium strategies.

Proposition 10. *In the LAB auction with n symmetric bidders each with demand for the two available goods determined by two independent draws from the distribution F where $F(0) > 0$, there is a unique symmetric equilibrium with differentiable bid functions satisfying equations* (14) *and* (15).

The restriction in Proposition 10 to choices of F with an atom at the lower endpoint can be replaced with other another assumption such as an assumption on the validity of using l'Hôpital's rule at the lower endpoint as in Bajari (2001). However, the discussion in Lebrun (2006) suggests that some additional assumption about the bidding behavior at the lower endpoint is needed to prove uniqueness in the first price auction. Given the close relation of our model to the first price uniqueness problem, we do not believe that we can prove uniqueness without such additional assumptions.

Placing an atom at the bottom of the distribution F is natural in many contexts.[58] For example the introduction of a reserve price r, however small, is equivalent to assuming a mass point at r; since our model is translation-invariant, this is essentially equivalent to a mass point at 0. Proposition 10 also guarantees uniqueness when aggregate demand is stochastic, if bidders potentially have zero value for items. Both cases maintain the close connection to the corresponding first price auction, in the well-studied cases of reserve prices or exogenous non-participation.

57 Lebrun (2006) points out that this assumption is implicit in Bajari (2001).

58 Our analysis in this paper assumes that F admits a continuous density everywhere on its support, however our results do not substantively change with the introduction of a mass point at the lower bound of the support.

D.2. Separation

We retain our focus on symmetric equilibria. Let $G_1(b)$ and $G_2(b)$ be the equilibrium distributions of *each bidder's* first and second bids. Define $F_{(k)}^{-1}$ as the inverse of the distribution of the kth order statistic from F, and let $\varphi_k = F_{(k)}^{-1} \circ G_k$ be the inverse bid function for unit k.[59] After isolating $g_1 \equiv dG_1/db$ and $g_2 \equiv dG_2/db$, when the bid monotonicity constraint does not bind the first order conditions require that

$$g_1(b) = \frac{1}{n-1}\left(\frac{2G_1(b)}{\varphi_2(b)-b}\right), \text{ and}$$
$$g_2(b) = \frac{G_2(b)-G_1(b)}{\varphi_1(b)-b} - 2\left(\frac{n-2}{n-1}\right)\left(\frac{G_2(b)-G_1(b)}{\varphi_2(b)-b}\right).$$

Since g_2 is a probability density, it must be that $g_2(b) \geq 0$. This requires that

$$\frac{1}{\varphi_1(b)-b} \geq 2\left(\frac{n-2}{n-1}\right)\frac{1}{\varphi_2(b)-b}.$$

Then $\varphi_2(b) \geq \varphi_1(b)$ when $n \geq 3$, and this inequality is strict when $n > 3$.[60] Note that the bid monotonicity constraint implies that $G_2(b) \geq G_1(b)$ for all b.

Proposition 11. *Suppose that there are n symmetric bidders, each with demand for $m_i = 2$ units, and $m = 2$ units are available for sale. Then in all symmetric equilibria of the LAB auction, bids are strictly separating.*

Proof. See Appendix B. □

The inequality at the heart of Proposition 11 exposes a distinction between models with and without market balance: when markets are unbalanced, maximum bids need not be equal across units. This is Corollary 5, which is useful in our examination of equilibrium uniqueness.

Corollary 5. *When $n > 3$ and equilibrium bids are monotone in values, $b_1(1) > b_2(1)$.*

With two bidders, a bid for unit k competes only against an opponent's bid for unit $m-k+1$; then distributions of bids for these units should have the same upper bound. With more agents this intuition fails. When there are only two units, for example, a bid for the second unit competes only against opponents' first-unit bids, while a bid for the first unit competes against opponents' bids for both the first and second units. Then it is no longer true that the bid distributions must be equal, only that the support of second-unit bids is a subset of the support of first-unit bids.

---

[59] The existence of a well-defined φ_k is formally established in the proof of Proposition 11. φ_k is well-defined as long as b_k is strictly monotone. If b_k is monotone but not strictly so, there are mass points in the distribution of market clearing prices, which implies a profitable deviation for some bidder.

[60] This implication is not bidirectional. Since market balance is satisfied when $n = 2$, we have already established that $\varphi_2(b) \geq \varphi_1(b)$ in this case.

References

Anderson, Edward J., Holmberg, Pär, Philpott, Andrew B., 2013. Mixed strategies in discriminatory divisible-good auctions. Rand J. Econ. 44 (1), 1–32.

Armantier, Olivier, Sbaï, Erwann, 2006. Estimation and comparison of treasury auction formats when bidders are asymmetric. J. Appl. Econom. (ISSN 1099-1255) 21 (6), 745–779. https://doi.org/10.1002/jae.875.

Armantier, Olivier, Sbaï, Erwann, 2009. Comparison of alternative payment mechanisms for French treasury auctions. Ann. Econ. Statist./Ann. Écon. Stat., 135–160.

Arozamena, Leandro, Cantillon, Estelle, 2004. Investment incentives in procurement auctions. Rev. Econ. Stud. 71, 1–18.

Ausubel, L.M., Cramton, P., Pycia, M., Rostek, M., Weretka, M., 2014. Demand reduction and inefficiency in multi-unit auctions. Rev. Econ. Stud. 81 (4), 1366–1400.

Back, Kerry, Zender, Jaime F., 1993. Auctions of divisible goods: on the rationale for the Treasury experiment. Rev. Financ. Stud. 6 (4), 733–764.

Bajari, Patrick, 2001. Comparing competition and collusion: a numerical approach. Econ. Theory 18, 187–205.

Blume, Andreas, Heidhues, Paul, 2004. All equilibria of the Vickrey auction. J. Econ. Theory 114 (1), 170–177.

Blume, Andreas, Heidhues, Paul, Lafky, Jonathan, Münster, Johannes, Zhang, Meixia, 2009. All equilibria of the multi-unit Vickrey auction. Games Econ. Behav. 66 (2), 729–741.

Bresky, Michal, 2013. Revenue and efficiency in multi-unit uniform-price auctions. Games Econ. Behav. 82, 205–217.

Burkett, Justin, Woodward, Kyle, 2018. Reserve Prices Eliminate Low Revenue Equilibria in Uniform Price Auctions.

Castellanos, Sara, Oviedo, Marco, 2008. Optimal bidding in the Mexican treasury securities primary auctions: results of a structural econometric approach. Cuad. Econ. 45 (131), 3–28.

Chassang, Sylvain, Ortner, Juan, 2015. Collusion in Auctions with Constrained Bids: Theory and Evidence from Public Procurement.

Chawla, Shuchi, Hartline, Jason D., 2013. Auctions with unique equilibria. In: Proceedings of the Fourteenth ACM Conference on Electronic Commerce. ACM, pp. 181–196.

Engelbrecht-Wiggans, Richard, Kahn, Charles M., 1998. Multi-unit auctions with uniform prices. Econ. Theory 12, 227–258.

Engelbrecht-Wiggans, Richard, Kahn, Charles M., 2002. Multiunit auctions in which almost every bid wins. South. Econ. J., 617–631.

Fevrier, Philippe, Préget, Raphaële, Visser, Michael, et al., 2002. Econometrics of Share Auctions.

Friedman, Milton, 1991. Single-price auction with multiple bids. Wall St. J.

Graham, Daniel A., Marshall, Robert C., 1987. Collusive bidder behavior at single-object second-price and English auctions. J. Polit. Econ. 95 (6), 1217–1239.

Häfner, Samuel, 2015. Value Bounds and Best Response Violations in Discriminatory Share Auctions.

Holmberg, Pär, 2009. Supply function equilibria of pay-as-bid auctions. J. Regul. Econ. 36 (2), 154–177.

Hortaçsu, Ali, Kastl, Jakub, 2012. Valuing dealers' informational advantage: a study of Canadian Treasury auctions. Econometrica 80 (6), 2511–2542.

Hortaçsu, Ali, McAdams, David, 2010. Mechanism choice and strategic bidding in divisible good auctions: an empirical analysis of the Turkish treasury auction market. J. Polit. Econ. 118 (5), 833–865.

Hortaçsu, Ali, Kastl, Jakub, Zhang, Allen, 2018. Bid shading and bidder surplus in the US treasury auction system. Am. Econ. Rev. 108 (1), 147–169.

Kang, Boo-Sung, Puller, Steven L., 2008. The effect of auction format on efficiency and revenue in divisible good auctions: a test using Korean Treasury auctions. J. Ind. Econ. 56 (2), 290–332.

Kirkegaard, René, 2009. Asymmetric first price auctions. J. Econ. Theory 144, 1617–1635.

Klemperer, Paul D., Meyer, Margaret A., 1989. Supply function equilibria in oligopoly under uncertainty. Econometrica 57 (6), 1243–1277.

Kremer, Ilan, Nyborg, Kjell G., 2004a. Divisible-good auctions: the role of allocation rules. Rand J. Econ., 147–159.

Kremer, Ilan, Nyborg, Kjell G., 2004b. Underpricing and market power in uniform price auctions. Rev. Financ. Stud. 17 (3), 849–877.

Kyle, Albert S., 1989. Informed speculation with imperfect competition. Rev. Econ. Stud. 56 (3), 317–355.

Lebrun, Bernard, 1999. First price auctions in the asymmetric n bidder case. Int. Econ. Rev. 40 (1), 125–142.

Lebrun, Bernard, 2006. Uniqueness of the equilibrium in first-price auctions. Games Econ. Behav. 55, 131–151.

Lebrun, Bernard, Tremblay, Marie-Christine, 2003. Multiunit pay-your-bid auction with one-dimensional multiunit demands. Int. Econ. Rev. 44 (3), 1135–1172.

LiCalzi, Marco, Pavan, Alessandro, 2005. Tilting the supply schedule to enhance competition in uniform-price auctions. Eur. Econ. Rev. 49 (1), 227–250.

Lizzeri, Alessandro, Persico, Nicola, 2000. Uniqueness and existence of equilibrium in auctions with a reserve price. Games Econ. Behav. 30 (1), 83–114.
Marszalec, Daniel, 2017. The impact of auction choice on revenue in treasury bill auctions-an empirical evaluation. Int. J. Ind. Organ.
Maskin, Eric, Riley, John, 2000. Asymmetric auctions. Rev. Econ. Stud. 67, 413–438.
Maskin, Eric, Riley, John, 2003. Uniqueness of equilibrium in sealed high-bid auctions. Games Econ. Behav. 45, 395–409.
McAdams, David, 2003. Isotone equilibrium in games of incomplete information. Econometrica 71 (4), 1191–1214.
McAdams, David, 2006. Monotone equilibrium in multi-unit auctions. Rev. Econ. Stud. 73 (4), 1039–1056.
McAdams, David, 2007a. Adjustable supply in uniform price auctions: non-commitment as a strategic tool. Econ. Lett. 95 (1), 48–53.
McAdams, David, 2007b. On the failure of monotonicity in uniform-price auctions. J. Econ. Theory 137 (1), 729–732.
McAdams, David, 2008. Partial identification and testable restrictions in multi-unit auctions. J. Econom. 146 (1), 74–85.
Milgrom, Paul Robert, 2004. Putting Auction Theory to Work. Cambridge University Press.
Myerson, Roger B., 1981. Optimal auction design. Math. Oper. Res. 6 (1), 58–73.
Plum, Michael, 1992. Characterization and computation of Nash-equilibria for auctions with incomplete information. Int. J. Game Theory 20 (4), 393–418.
Pycia, Marek, Woodward, Kyle, 2019. Auctions of Homogeneous Goods: A Case for Pay-as-Bid.
Reny, Philip J., 2011. On the existence of monotone pure-strategy equilibria in Bayesian games. Econometrica 79 (2), 499–553.
Vickrey, William, 1961. Counterspeculation, auctions, and competitive sealed tenders. J. Finance 16 (1), 8–37.
Vives, Xavier, 2011. Strategic supply function competition with private information. Econometrica 79 (6), 1919–1966.
Wang, James J.D., Zender, Jaime F., 2002. Auctioning divisible goods. Econ. Theory 19 (4), 673–705.
Wilson, Robert, 1979. Auctions of shares. Q. J. Econ., 675–689.
Woodward, Kyle, 2016. Strategic Ironing in Pay-as-Bid Auctions.
Woodward, Kyle, 2017. Equilibrium and Approximation in Auction-Like Models.
Woodward, Kyle, 2019. Hybrid mechanisms for the sale of divisible goods. Spinger.