Sharing Cost Information in Dynamic Oligopoly*

Greg Kubitz† and Kyle Woodward‡

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Abstract

We study the effect of sharing cost information in dynamic oligopoly. Firms can agree to verifiably share information about common costs, as with the aggregation of input costs by an industry trade association. Cost information that is not directly shared is revealed through observed prices. We show that such information sharing agreements lead to higher prices and reduce consumer surplus when either demand is inelastic or goods are highly substitutable. Information sharing agreements that increase the equilibrium informativeness of prices increase expected prices and reduce consumer surplus. In markets with a large number of firms, information sharing has a minimal impact on expected prices and can increase both consumer and producer surplus when goods are not too substitutable.

1 Introduction

In a strategic market interactions, prices contain information about firms' underlying cost structures. What a competitor learns about a firm's costs depends on what it knows prior to observing the firm's price. This dependence is both direct — better initial information leads to better inference from prices — and indirect — changes in the initial information structure lead to changes in pricing strategies. A competitor's ability to learn from a firm's prices affects their own pricing strategies, and the initial information structure therefore affects market surplus. In this paper we consider the effect of sharing information about

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†Queensland University of Technology; gregory.kubitz@qut.edu.au

‡University of North Carolina at Chapel Hill; kyle.woodward@unc.edu

common costs while hiding information about *specific* costs, in the context of dynamic price competition. We show that information sharing unambiguously increases prices, but welfare effects depend on the substitutability of goods and the elasticity of demand.

In practice, information sharing is one key service of trade associations. Trade associations can increase producer surplus by aggregating information regarding demand or costs, or by explicitly coordinating strategic decisions.[1] Improved producer surplus frequently comes at the cost of consumer surplus, and price coordination and private information sharing are generally considered anti-competitive by antitrust authorities.[2] An "honest" trade association, barred from sharing strategic plans or firm-specific information, may still share information about industry trends. For example, it may generate a market forecast of raw input prices. We model this honest trade association, which aggregates only information about costs which are shared by all firms, and we show that when goods are substitutable or demand is inelastic even this seemingly innocuous trade association will be anti-competitive.

Our main contribution is the identification of a novel and endogenous inference channel which is affected by information sharing. This channel arises from the separation of firm costs into common and specific components. As discussed above, an honest trade association is restricted to sharing industry-wide information, and firms within a trade association may remain uncertain of each others' costs. In this context, sharing information provides increased precision regarding common costs, which in turn increases the incentive to soften subsequent competition. This tradeoff cannot exist without multidimensional uncertainty. Our analysis assumes a multidimensional cost structure (vs. e.g., Sweeting et al. [2019]), perfect observations of market outcomes (vs. e.g., Bonatti et al. [2017]), and inference from these observations (vs. e.g., Raith [1996]); we review related literature in detail below. We show that expected prices are monotonically related to the informativeness of prices, and use this relationship to derive results on the welfare effects of information sharing. In an extension, we show that all our results are robust to partial verifiability.[3]

Before a further overview of our results we give a basic statement of our model. We study

[1]Examples of the former include *Case AT.40136 — Capacitors*, and *American Column & Lumber Co. v. United States*. Examples of the latter include *United States v. Nippon Paper Indus. Co., Ltd*, and *United States v. Archer Daniels Midland Co.*. For an overview of different forms of information sharing, see Kühn [2001] and Marshall and Marx [2014].

[2]The extent to which information sharing is permissible depends on the antitrust authority. In the U.S., regulators judge information sharing agreements between competitors by the *rule of reason* but note "[...] the sharing of information relating to price, output, costs, or strategic planning is more likely to raise competitive concern than [...] less competitively sensitive variables" (April 2000 FTC/DOJ Guildelines for Collaborations Among Competitors). In Europe, regulators have stricter principles declaring that sharing of information relevant to future prices is a restriction of competition by object (2011 Guidelines for Article 101 of TFEU).

[3]In Section 5 we model partial verifiability as firms sharing only a fraction of their cost information. This is equivalent to verifiability of only some dimensions of a multidimensional cost structure.

dynamic differentiated-good Bertrand competition where demand is common knowledge but firms have private information about costs.[4] Costs consist of two components: specific costs incurred by a particular firm (for example, a labor contract), and common costs that are shared by all firms in an industry (for example, raw input prices). In the first period each firm has imperfect information regarding its own costs, as well as those of its opponents. Firms may share verifiable information about common costs, as through a trade association. If they do so, they will have identical information about common costs but remain uninformed about their competitors' private costs.[5] Before second period competition each firm learns its own costs. To infer the costs of competitors, firms interpret first period prices as signals of costs, and update their beliefs accordingly before making second period pricing decisions.

We give an implicit characterization of equilibrium in linear pricing strategies and show that it is unique (Theorem 1). In this equilibrium we analyze the effects of sharing information about common costs. Sharing information increases the precision of firms' beliefs. The immediate effect is that firms can set prices closer to the known-cost optimum, which increases producer surplus. Tying pricing decisions more tightly to actual costs increases the variance of each firm's price as well as the covariance of the firms' prices. Increased price variance improves ex ante consumer surplus, while the effect of covariance depends on the substitutability of the firms' products.[6]

In addition to the direct effects of improved precision, information sharing indirectly impacts market outcomes by altering pricing strategies. If common cost information is not shared, firms have two-dimensional private information: signals of their specific costs, and signals of their common costs. As illustrated in Figure 1, first period prices pool information, and a high first period price may be indicative of high expected specific costs, high expected common costs, or both. If common cost information is shared each firm's private information is reduced to a single dimension, and its first period price will be more informative of its specific costs.[7] Irrespective of information sharing, firms have an incentive to increase first period prices in hopes of indicating high costs and softening subsequent competition. When

[4] Analyzing Cournot competition in our framework is straightforward but tedious. Roughly, Cournot competition reverses the signs of our results under Bertrand competition; this is in line with, e.g., Vives [1984] and Raith [1996].

[5] In Section 5 we adapt a model from Vives [2001] to analyze the case of partial information sharing, where firms share verifiable but imperfect signals of their own information about common costs. Under partial information sharing, firms do not have identical information about common costs.

[6] This same logic is at play in the rich literature on sharing information about uncertain demand. See the early work of Clarke [1983], Vives [1984], and Kirby [1988], among others. These results focus on static competition and do not consider dynamic incentives to hide private information.

[7] In the separating equilibrium we obtain, when common cost information is shared the firm's first period price is perfectly indicative of its first period signal of specific costs. Because the initial signal of specific costs is noisy, its competitor remains uncertain of the firm's specific costs even after witnessing first period prices.

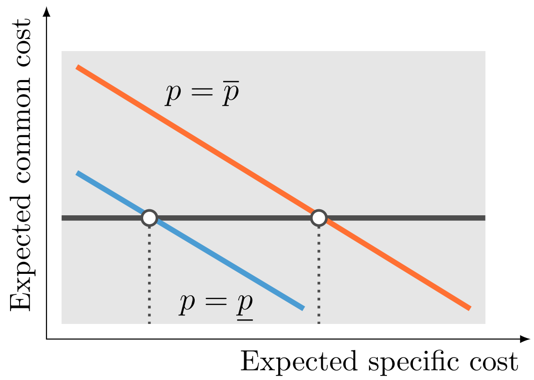

Figure 1: A high price (orange line) can be indicative of a high expected common cost or a high expected specific cost, while a low price (blue line) can be indicative of a low expected common cost or a low expected specific cost. When the opponent believes all signals are possible (shaded area) its information after witnessing the firm's price is that signals fell somewhere on the iso-price curve. When the opponent has full knowledge of the firm's expectation of common costs (dark gray line) it can uniquely identify the firm's expected specific cost (dotted lines).

prices are more informative about firm-specific costs firms have a stronger incentive to over-represent these costs, and prices are more distorted when information is shared. Thus, as we show, information sharing increases expected first period prices (Theorem 3).[8]

The overall effect of information sharing on welfare is determined by the relative impacts of increased precision and incentives to soften competition (i.e., increased prices). We consider three cases. First, when demand is relatively inelastic, strategic effects dominate. Demand does not respond strongly to the increase in prices caused by information sharing, and information sharing reduces consumer surplus. The effect on producer surplus depends on substititability: information sharing increases producer surplus when goods are substitutes and decreases producer surplus when goods are complements. Second, when goods are relatively substitutable (independent of demand elasticity) the price increase induced by information sharing outweighs any benefits of increased variance as well as the increased covariance of prices (Proposition 4). Third, when the market is large strategic considerations essentially vanish.[9] In this case, information sharing always improves producer surplus (Proposition 5), while its effect on consumer surplus depends on the substituability of goods and the initial information about common costs (Proposition 6). These effects are summa-

[8] As we also show, information sharing has no effect on expected second period prices. All welfare effects in the second period arise from the increased ability of firms to learn about their opponents' specific costs when common cost information is shared.

[9] The bulk of our analysis is carried out in the context of a duopoly. We show that the basic structure of equilibrium applies to any number of firms and provide a welfare analysis as the number of firms goes to infinity.

	Complements ($r < 0$)	Mild substitutes ($0 < r < 1/2$)	Strong substitutes ($1/2 < r$)
Inelastic demand[†] ($a \gg b$)	PS↓, CS↓	PS↑, CS↓	PS↑, CS↓
Large market ($n \to \infty$)	PS↑, CS↑	PS↑, CS↑[⋆]	PS↑, CS↓

[†] Inelastic demand assumes $n = 2$ firms (c.f. Proposition 4).

[⋆] Large market consumer surplus with mild substitutes assumes initial precision is low (c.f. Proposition 6).

Table 1: The surplus effects of information sharing. Notation given in Section 2.

rized in Table 1.

Finally, we consider the impact of general information sharing agreements, where firms agree to share a portion of their initial information. This type of agreement may arise in the case where only some cost information is verifiable. We show that expected prices are ranked by equilibrium informativeness of prices (Proposition 7). Moreover, sharing any amount of information about common costs increases price informativeness, and therefore increases expected prices; on the other hand, sharing information about firm specific costs reduces price informativeness, and therefore reduces expected prices (Proposition 8). Because our surplus results depend only on the informativeness of prices, increased sharing of common cost information still reduces consumer surplus, while increased sharing of specific cost information increases consumer surplus, provided demand is relatively inelastic (Corollary 3). A natural interpretation is that a trade association interested in maximizing producer surplus should aggregate and share as much (or as little) information as possible: all solutions to this trade association's problem are corner solutions.[10]

While motivated by information sharing agreements that are facilitated by trade associations, our results are not sensitive to the source of aggregated information. Consumer surplus can be harmed by any public source of common cost information. For example, firms may reduce development costs by eliminating in-house research teams and outsourcing market research to a consultant. If this consultant is used by all firms in an industry (where goods are substitutes) consumer surplus will be lower than if firms generate market forecasts independently. On the other hand, our results suggest that consumers may be better off if many firms across complementary industries share a common source of input market data.

[10]In particular, we do not consider the objective function of the entity responsible for aggregating and disseminating shared information. A trade association partially interested in, e.g., consumer surplus, may choose to aggregate less common cost information.

1.1 Related literature

Our results relate to past work on the competitive effects of information sharing, and the dynamic revelation of information. Two properties of our model distinguish our results from the existing literature. First, firms have multidimensional private information about costs, which are partially common. Second, competition is dynamic, so observed prices signal information that affects subsequent competition.

In dynamic oligopoly models with incomplete information firms distort strategies to signal information that is beneficial to the firm in later stages of competition. In dynamic Bertrand competition with private information about costs, Mailath [1989], Mester [1992], and more recently Sweeting et al. [2019] describe the incentive to soften competition by over-representing cost through a choice of a price that is higher than is stage optimal. In Cournot competition with observed prices, firms can signal jam when selecting unobserved output quantities. Mirman et al. [1993] look at the case where firms have private information about individual demand curves. Bonatti et al. [2017] characterize the dynamics of signal jamming and learning when firms begin with private information about (only) specific costs. In our Bertrand framework, firms have the familiar incentive to soften competition by over-representing costs, and a rich strategy space by which to achieve this goal. Signal jamming relates to the weight the firm places on each source of information when choosing price. By reducing the weight on one source of information the firm reduces the informativeness of the price about this source.[11] In settings other than dynamic oligopoly, the impact of external incentives on the informativeness of the signal given two dimensions of private information has been considered in Frankel and Kartik [2019], Bénabou and Tirole [2006], Fischer and Verrecchia [2000], and Bagwell [2007], among many others.

In models with specific and common costs, firms put different weights on information about different cost components. In this paper, firms weigh common cost information more than specific cost information when selling substitutes, and weigh specific cost information more than common cost information when selling complements. This follows the basic intuition of Angeletos and Pavan [2007] where agents have both private and public signals about a single parameter and weigh private (public) signals more when actions are strategic substitutes (complements).[12] Similarly, when firms compete in supply schedules, Bernhardt and Taub [2015] shows that firms will place additional weight on private signals over common signals. The ability to signal jam in their setting increases the difference in the relative use

[11] Mirman et al. [1994] identify that an increase in the choice of quantity will reduce the informativeness of price about uncertain demand in the Cournot setting where incomplete information is symmetric.

[12] In a Bertrand setting, Myatt and Wallace [2015b] show that too much public information is used from the perspective of consumers, and a less-than-efficient amount is used from the perspective of the firms. Myatt and Wallace [2015a] find opposite results in Cournot competition.

of information, as firms prefer to reduce the informativeness of market prices about privately observed, common-valued information.

Results are mixed concerning the impact of information sharing on consumer welfare in oligopoly competition with private cost information. This has lead to differing conclusions about the competitive nature of information sharing agreements; see discussions in Kühn and Vives [1995] and Vives [2001]. Under monopolistic competition, Vives [1990] shows that information sharing harms total surplus under price setting while improving it under quantity setting. Under Bertrand competition, prices are strategic complements and information sharing increases the covariance of prices, leading to larger variance of quantity and lower expected surplus. Under Cournot competition, quantities are strategic substitutes and information sharing leads to lower variance of aggregate quantity, and therefore higher expected surplus.[13] In our setting expected prices increase when information is shared, and this reduces the likelihood that information sharing yields a welfare improvement. When the number of firms is large, signal jamming incentives are minimized and welfare improvement is possible: both profits and consumer surplus increase when there is significant differentiation between products.[14] Additionally, incentives to share private cost information in Bertrand competition depend on the structure of firms' information. For example, firms may prefer to share no information about perfectly-known private costs [Gal-Or, 1986], but it may be profitable to share affiliated noisy signals of cost parameters [Sakai, 1986].[15]

Finally, in a similarly-motivated paper Jeitschko et al. [2018] examine the impact of firms sharing information about a private-valued cost component in the context dynamic competition. When costs are one-dimensional, information sharing eliminates all private information, and along with it all incentive to soften competition. While firms directly benefit from the increased precision of shared information, the strategic effects of more precise information reduce expected prices and information sharing is not generally profitable. Our paper offers a stark comparison: sharing common cost information *increases* the incentive to soften competition and can therefore have a positive effect on profits. We identify that the qualitative difference in these results stems from the differing impact of the information sharing agreement on the equilibrium informativeness of price as a signal of private costs.

The rest of the paper is organized as follows. Sections 2 and 3, respectively, introduce and analyze the model of two period price competition without information sharing. Section 4

[13] In Cournot competition, Shapiro [1986] shows that information sharing of cost information increases total welfare and Sakai and Yamato [1989] show consumer surplus improves when products are differentiated enough and costs are positively correlated, similar to our conditions for an increase in consumer welfare.

[14] In the case of procurement auctions, Asker et al. [2019] show that sharing information about bidder competitiveness can improve bidder and consumer surplus at the expense of auctioneer revenue.

[15] See further discussions in Raith [1996] and Vives [2001].

studies the strategic and welfare impact of information sharing in a duopoly and in the case when the number of firms is large. In Section 5 we consider the impact of more general information sharing agreements. Section 6 concludes. Most proofs are given in the appendix.

2 Model

Two firms, i and j, compete for market share over two periods $t \in \{1, 2\}$. Demand is linear in prices, symmetric across firms, and time-independent. Firm i's demand is given by

$$q_{it} = a - bp_{it} + ep_{jt}.$$[16]

The demand parameters a and b are strictly positive, while the sign of e determines whether goods are complements ($e < 0$) or subsitutes ($e > 0$). We assume that demand is weakly more sensitive to a firm's own price than to its opponent's, so that $-b \leq e \leq b$. Throughout, we let $r \equiv e/b$, $-1 \leq r \leq 1$, denote the relative dependence of firm i's demand on firm j's price. Each firm faces a constant marginal cost c_i that is the same in each period, so profits are

$$\pi_{it} = (p_{it} - c_i)\, q_{it}.$$

Firms are initially uncertain about their marginal costs of production, but know that costs are comprised of a specific component θ_i and a common component ρ; firm i's constant marginal cost is the sum of the two components, $c_i = \rho + \theta_i$. We assume that cost components are joint-normally distributed with zero covariance, so that $\theta_i, \theta_j \sim N(\mu_\theta, \sigma_\theta^2)$ and $\rho \sim N(\mu_\rho, \sigma_\rho^2)$. We further assume that demand is sufficiently strong relative to costs and elasticity, $a \geq (b - e)\mathbb{E}[c_i]$. Throughout, we will denote the precision of θ_i, θ_j by $\tau_\theta = 1/\sigma_\theta^2$ and the precision of ρ by $\tau_\rho = 1/\sigma_\rho^2$.

Play proceeds in two periods. In the first period, each firm receives two noisy signals, $s_{i\theta}$ and $s_{i\rho}$, of the values of their specific and common costs, respectively.[17] These signals are normally distributed with uncorrelated error terms, and the error terms are uncorrelated between firms. We model these signals as $s_{i\theta} = \theta_i + \varepsilon_{i\theta}$ and $s_{i\rho} = \rho + \varepsilon_{i\rho}$, where $\varepsilon_{i\theta}$ and $\varepsilon_{i\rho}$ are independent and normally distributed with mean zero and variance $\sigma_{i\theta}^2$ and $\sigma_{i\rho}^2$, respectively; we denote the relative precision of firm i's signal of variable x by $\overline{\tau}_{ix} = \tau_{ix}/(\tau_x + \tau_{ix}) = \sigma_x^2/(\sigma_x^2 + \sigma_{ix}^2)$, and assume parameters are such that $\overline{\tau}_{ix} \in (0, 1)$.[18] All uncertainty is common

[16]Unless otherwise specified, our equations and inequalities should be taken to be symmetric for agent j.

[17]It is not essential that firm i has imperfect information regarding its specific cost θ_i, since our results depend only on what firm j can learn about firm i's specific costs. We assume noisy signals of specific costs for consistency with noisy signals of common costs; we discuss this assumption further in Section 3.

[18]Our algebraic results are unaffected by the assumption that $\overline{\tau}_{ix} \notin \{0, 1\}$ (taking limits where appro-

knowledge.

In our benchmark model, without information sharing, firms simultaneously select prices p_{i1}, p_{j1} immediately after private information is realized. We later (in Section 4) allow firms to share verifiable information about the common cost component ρ.[19] In this case, firms simultaneously select prices immediately after information is shared. Regardless of the information sharing regime, firm i observes stage profits π_{i1} and its competitor's price p_{j1} immediately after prices are set. Firms then learn both the common and their (individual) specific cost components, but remain unaware of their opponent's specific cost component.[20] Firms then compete in a second period by simultaneously selecting prices and obtain stage profits π_{i2}.

The game ends after the second period, and ex post profits are the (undiscounted) sum of stage profits,

$$\pi_i\left(p_i, p_j\right) = \pi_{i1}\left(p_{i1}, p_{j1}\right) + \pi_{i2}\left(p_{i2}, p_{j2}\right).$$

We restrict attention to *perfect Bayesian equilibria in linear strategies.*

Definition 1. *A* linear strategy *is given by parameters* $(p_{it0}, p_{it\theta_i}, p_{it\theta_j}, p_{it\rho})_{t\in\{1,2\}}$*, such that firm i's price in period t given history* h_{it} *is:*

$$p_{it}(h_{it}) = p_{it0} + p_{it\theta_i}\mathbb{E}\left[\theta_i | h_{it}\right] + p_{it\rho}\mathbb{E}\left[\rho | h_{it}\right] + p_{it\theta_j}\mathbb{E}\left[\theta_j | h_{it}\right].$$

Definition 2. *A* perfect Bayesian equilibrium in linear strategies *is a set of linear strategies, one for each firm, such that*

1. *Second period prices maximize profits given any history* $h_{i2} = (s_{i\theta}, s_{i\rho}, \theta_i, \rho, \mathbf{p}_1)$*:*

$$\begin{aligned} p_{i2}(h_{i2}) &= p_{i20} + p_{i2\theta_i}\theta_i + p_{i2\rho}\rho + p_{i2\theta_j}\mathbb{E}\left[\theta_j | \rho, \mathbf{p}_1\right] \\ &\in \underset{\tilde{p}}{\operatorname{argmax}}\, \mathbb{E}\left[\left(a - b\tilde{p} + ep_{j2}\right)\left(\tilde{p} - \left[\theta_i + \rho\right]\right) | h_{i2}\right]; \end{aligned}$$

priate). However, our surplus results depend on partial strict informativeness of signals. For example, if $\overline{\tau}_{i\rho} = 0$, the signal $s_{i\rho}$ of the common cost component contains no additional information, and aggregating common cost information has no effect; the same is true when $\overline{\tau}_{i\rho} = 1$ and the signal $s_{i\rho}$ of common costs are perfectly informative.

[19] In Section 5 we consider the possibility that firms share partial, but still verifiable, information about specific and/or common costs.

[20] Since demand is a deterministic function of firm prices, the assumption that firms witness their own profits and each others' prices is sufficient to imply that they are perfectly informed of their own private cost c_i. Alternatively, if they witness their own sales volume they will be perfectly aware of their opponent's price. That they obtain perfect knowledge of each of the components of $c_i = \rho + \theta_i$ is an additional assumption.

2. *First period prices maximize profits given history $h_{i1} = (s_{i\theta}, s_{i\rho})$:*

$$p_{i1}(h_{i1}) = p_{i10} + p_{i1\theta_i}\mathbb{E}\left[\theta|\, s_{i\theta}, s_{i\rho}\right] + p_{i1\rho}\mathbb{E}\left[\rho|\, s_{i\theta}, s_{i\rho}\right]$$
$$\in \underset{\tilde{p}}{\operatorname{argmax}}\, \mathbb{E}\left[(a - b\tilde{p} + ep_{j1})\left(\tilde{p} - [\theta_i + \rho]\right) + \pi_{i2}\left(p_{i2}, p_{j2}\right)|\, s_i\right];$$

3. $p_{i1\theta_j} = 0$;[21]

4. *Strategies are symmetric:*

$$(p_{it0}, p_{it\theta_i}, p_{it\theta_j}, p_{it\rho})_{t\in\{1,2\}} = (p_{jt0}, p_{jt\theta_j}, p_{jt\theta_i}, p_{jt\rho})_{t\in\{1,2\}}.$$

In an equilibrium in linear strategies, prices are an affine function of expected common and specific costs. The equilibrium we find is without constraint to symmetric linear strategies, but we do not address the potential existence of equilibria in asymmetric or nonlinear strategies.

Two expositional notes are in order. First, following our initial equilibrium analysis, we consider an informational regime in which firms share their signals of the common cost ρ. Throughout, we use variables decorated with * (e.g., $\pi^\star$) to indicate values in the no-sharing equilibrium, and we use variables decorated with ^{c} (e.g., π^c) to indicate values in the equilibrium which arises following the sharing of common cost information. Second, for most of our analysis we focus on symmetric equilibria, and on the effects of information sharing on first period prices.[22] For space and (we hope) legibility we therefore abbreviate $p_{i10} \equiv p_0$, $p_{i1\theta} \equiv p_\theta$, and $p_{i1\rho} \equiv p_\rho$, where we do not believe it will create confusion.

3 Equilibrium

We compute equilibrium in the two period model by backwards induction.[23] In the second period, each firm knows its own marginal costs exactly, but has a distribution representing its beliefs over its opponent's costs. Letting $\mathbf{p_1} \equiv (p_{i1}, p_{j1})$ and $F^j(\cdot;\rho,\mathbf{p}_1) \equiv F^j$ be the distribution of firm j's second period price conditional on firm i's available information,[24]

[21] This constraint exists because, in our benchmark model, $\mathbb{E}[\theta_j|s_i] = \mu_\theta$ is constant, hence p_{i10} cannot be identified from $p_{i1\theta_j}$. We consider partial information about opponent specific cost θ_j in Section 5, and accordingly allow $p_{i1\theta_j} \neq 0$.

[22] As we show, expected second period prices are unaffected by the information sharing regime.

[23] It is straightforward to show that, in a linear equilibrium, prices must respond to information. Since signals are normally distributed, all prices p_{i1} are on-path, and there is no need to consider off-path behavior.

[24] Firm i also knows θ_i, $s_{i\theta}$, and $s_{i\rho}$, but these offer no information in the second period about firm j's pricing strategy. Equivalently, $(\rho, \mathbf{p}_1)$ is firm i's knowledge of information common to both firms.

the profit maximization problem is

$$\max_p \int (p - c_i)(a - bp + ex)\, dF^j(x; \rho, \mathbf{p}_1).$$

Lemma 1. *Firm i's optimal second period price is*

$$p_{i2}^\star = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right]\right).$$

Firm i's maximum second period expected profit is

$$\mathbb{E}\left[\pi_{i2}^\star \middle| \rho, \mathbf{p}_1\right] = \frac{1}{4b}\left(a - bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right]\right)^2.$$

Because the game ends in the second period, optimal second period prices are identical to those in a single period duopoly model where the opponent's price is distributed according to $F^j(\cdot; \rho, \mathbf{p}_1)$. Firm i's second period price is an affine function of the demand intercept, its (known) cost $c_i = \rho + \theta_i$, and its expectation over firm j's second period price. Profits then have a standard quadratic form.

Lemma 2. *In any equilibrium, expected second period prices of a firm given publicly available information are*

$$\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right] = \frac{1}{4b^2 - e^2}\left((2b + e)\, a + 2b^2\mathbb{E}\left[c_j \middle| \rho, p_{j1}\right] + be\mathbb{E}\left[c_i \middle| \rho, p_{i1}\right]\right),$$

which result in the following expected second period profits:

$$\mathbb{E}\left[\pi_{i2}^\star \middle| \rho, \mathbf{p}_1\right] = \frac{1}{4}\left(\frac{1}{4b^2 - e^2}\right)^2\left((4b + 2e)\, a - 4b^2 c_i + \left(\mathbb{E}\left[c_i \middle| \rho, p_{i1}\right] - c_i\right) e^2 + 2be\mathbb{E}\left[c_j \middle| \rho, p_{j1}\right]\right)^2.$$

Note that the expression for expected profits in Lemma 2 is written in terms of expected costs conditional only on the information relevant to forming an expectation of each firm's costs. Although firm i's first period price p_{i1} is informative regarding the common cost ρ and may be useful to make inferences from firm j's price p_{j1}, once in the second period firms have full knowledge of ρ and firm i's first period price p_{i1} no longer provides further information regarding firm j's specific cost θ_j.

3.1 First period pricing

First period prices are set to optimize the sum of profits over two periods. Although first period prices have no direct effect on second period profits, firm i's price affects firm j's beliefs regarding firm i's costs. This is apparent from Lemma 2, where p_{i1} enters only through $\mathbb{E}[c_i|\rho, p_{i1}]$, which is the expectation of firm i's cost given information available to firm j in the second period. Firm i has an incentive to over-represent its cost, leading firm j to increase its second period price, softening competition for firm i.[25] The first period profit maximization problem is

$$\max_p \mathbb{E}\left[(a - bp + ep_{j1})(p - c_i) + \pi^\star_{i2} \middle| s_{i\rho}, s_{i\theta}\right].$$

A marginal increase in first period price affects first period profits in a standard way, and has an additional effect on second period profits by manipulation of the opposing firm's second period beliefs. The first order condition is given in Lemma 3.

Lemma 3. *For a given pricing strategy of firm j, firm i's optimal first period price is*

$$\begin{aligned}\hat{p}_{i1} = &\left(\frac{1}{2b}\right)\mathbb{E}\left[bc_i + a + ep_{j1}\middle| s_{i\rho}, s_{i\theta}\right] \\ &+ e\left(\frac{1}{2b}\right)^2 \mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p^\star_{j2}\middle|\rho, \hat{p}_{i1}, p_{j1}\right]\right)\frac{\partial}{\partial p_{i1}}\mathbb{E}\left[p^\star_{j2}\middle|\rho, \hat{p}_{i1}, p_{j1}\right]\middle| s_{i\rho}, s_{i\theta}\right].\end{aligned}$$

Under linear strategies, each firm's first period price choice is a normally distributed random variable from the perspective its opponent. Therefore, (c_i, ρ, p_{i1}) are distributed joint-normally and $\mathbb{E}[c_i|\rho, p_{i1}]$ is linear in p_{i1}. Additionally, the effect of an increase in firm i's first period price on firm j's second period beliefs, and hence second period price, is constant and independent of the level of price. Conditioning beliefs on this relationship gives Lemma 4.

Lemma 4. *The marginal effect of firm i's first period price on firm j's expected second period price is*

$$\frac{\partial}{\partial p_{i1}}\mathbb{E}\left[p^\star_{j2}\middle|\rho, p_{i1}\right] = \frac{r}{4 - r^2}\kappa_i,$$

$$\textit{where } \kappa_i \equiv \frac{\partial}{\partial p_{i1}}\mathbb{E}\left[c_i\middle|\rho, p_{i1}\right] = \frac{\sigma^2_\theta \bar{\tau}_{i\theta} p_{i\theta}}{\sigma^2_\rho(1 - \bar{\tau}_{i\rho})\bar{\tau}_{i\rho}p^2_{i\rho} + \sigma^2_\theta\bar{\tau}_{i\theta}p^2_{i\theta}}.$$

The form of the expressions in Lemma 4 makes clear that the response of firm j's second period price to firm i's first period price depends only on the relative substitutability (or complementarity) r of the two firms' goods and the informativeness of firm i's first period

[25] This describes the reaction of firm j when the firms are selling substitutes, $e > 0$. When goods are complements, $e < 0$, a higher value of $\mathbb{E}[c_i|\rho, p_{i1}]$ leads to a lower $p^\star_{j2}$, which still increases $\pi^\star_{i2}$. When $e = 0$, the price and profit equations reduce to the standard monopoly model.

price. Specifically, the term κ_i captures the relative informativeness of firm i's first period price regarding its specific cost component θ_i, the remaining source of asymmetric information in the second period once the common cost ρ is commonly known. Despite observing ρ, firms do not observe each other's first period signal of the common cost component, $s_{i\rho}$. Because the first period price depends on the realization of $s_{i\rho}$, price is a noisy signal of $s_{i\theta}$. Therefore the informativeness of the price with respect to θ_i depends not only on the variance of the price relative to $s_{i\theta}$ but also relative to $s_{i\rho}$.

The choice of strategy in the first period for a given level of precision directly impacts the value of κ_i. Specifically, κ_i decreases as either $p_{i\theta}$ or $p_{i\rho}$ increases. If coefficient p_{ix} increases while signal precisions remain constant, the variance of price increases, and therefore changes in price will be less informative of the firm's private information. Moreover, the incentive constraints on the equilibrium strategy in the first period depend on the value of κ_i.[26] This fixed point problem is expressed in the single variable equation in Theorem 1.

Theorem 1. *There exists a unique symmetric equilibrium in linear pricing strategies. The equilibrium strategies are determined by the value of κ in equilibrium which satisfies the following single variable equation:*

$$\kappa^\star = \frac{\sigma_\theta^2 \bar{\tau}_\theta p_\theta^\star}{\sigma_\rho^2 \left(1-\bar{\tau}_\rho\right) \bar{\tau}_\rho p_\rho^{\star 2} + \sigma_\theta^2 \bar{\tau}_\theta p_\theta^{\star 2}},$$
$$\textit{subject to } p_\theta^\star = \frac{1}{2+\beta\kappa^\star} \textit{ and } p_\rho^\star = \frac{1-\left(\frac{1-r}{2-r}\right)\beta\kappa^\star}{2-r\bar{\tau}_\rho - \frac{1}{2}\left(1-\bar{\tau}_\rho\right)\beta^2\kappa^{\star 2}},$$

where $\beta = r^2/(4-r^2)$.

As is the case with second period prices, firms' equilibrium responses to information depend only on the relative substitutability (or complementarity) of their goods, and not the magnitude of demand response to prices.[27]

Remark 1. *With the exception of comparative statics on $\bar{\tau}_\theta$, our results are essentially unaffected by the assumption that firm i has imperfect information of its specific costs θ_i. Letting firms have perfect knowledge of their specific costs while remaining uncertain of their opponent's costs is equivalent to letting $\bar{\tau}_\theta = 1$ while $\tau_\theta < \infty$. In this case firm j remains uncertain of firm i's specific costs, and firm i's incentive to soften future competition is qualitatively unchanged. As a modeling assumption, we retain imperfect information regarding specific costs for consistency with imperfect information regarding common costs.*

[26] Since pricing strategies are not observed, κ_i is not affected by firm i's selection of price; it is determined by the pricing strategy the firm is believed to be following.

[27] The absolute magnitudes of b and e will factor in to p_0, the additive component of the linear price structure.

Our algebraic results also remain valid when specific costs are common knowledge ($\sigma_\theta^2 = 0$). In this case, there is no private information in the second round of competition, hence there is no incentive to soften future competition. Equilibrium price coeffecients are standard, $p_\theta^\star = 1/2$ and $p_\rho^\star = 1/(2 - r\overline{\tau}_\rho)$.

There are two strategic effects we can identify in first period prices. First, due to the correlation of one cost signal and the independence of the other signal, firms may want to act more heavily on one of these signals than the other if they prefer to have their prices correlated in the first period. Additionally, firms benefit from having private information in the second period and therefore prefer to not reveal precise information about their specific cost term. The implications of the first effect are in Proposition 1 and the implications of the second effect are in Proposition 2.

Proposition 1. *When goods are complements ($e < 0$), $p_\rho^\star < p_\theta^\star$; when goods are substitutes ($e > 0$), $p_\rho^\star > p_\theta^\star$; when goods are independent ($e = 0$), $p_\rho^\star = p_\theta^\star$.*

When $e > 0$, so that goods are substitutes, firms' first period prices are more sensitive to information about the common cost component than to information about their specific cost components. If a firm receives a high signal on the common cost component this typically implies the other firm will set a high price, increasing demand and making it optimal to further increase price. When $e < 0$, so that goods are complements, prices are strategic substitutes and will not respond strongly to the common cost signal. When $e = 0$, so that there are no cross-firm demand effects, there is no need to adjust for the opponent's price or, correspondingly, to conceal information regarding cost, and therefore information about each cost component affects first period prices identically.

Proposition 2. *The values of $p_\theta^\star$ and $\kappa^\star$ are inversely related: $p_\theta^\star$ increases when $\kappa^\star$ decreases and vice versa. Additionally, $p_\theta^\star$ is decreasing and $\kappa^\star$ is increasing in $\overline{\tau}_\theta$, and there is a $\hat{\tau}$ such that for all $\overline{\tau}_\rho > \hat{\tau}$, $\kappa^\star$ is increasing and $p_\theta^\star$ is decreasing in $\overline{\tau}_\rho$, and for all $\overline{\tau}_\rho < \hat{\tau}$, $\kappa^\star$ is decreasing and $p_\theta^\star$ is increasing in $\overline{\tau}_\rho$. When $e > 0$, $\hat{\tau} > 1/2$ and when $e < 0$, $\hat{\tau} < 1/2$.*

The presence of uncertainty regarding the common cost component adds noise to the relationship between first period price and the specific cost signal. When this relationship is noisier, the price reveals less information about this signal, allowing the firm to use available information in its pricing decision without revealing its actual costs. If the signal about the common cost is relatively imprecise ($\overline{\tau}_\rho \approx 0$) then firms do not learn much from this signal, and relatively little noise is added to this relationship. Additionally, if the signal is very precise ($\overline{\tau}_\rho \approx 1$) then when firms learn the true value of ρ in the second round, they will learn with little error what signal their opponents received and will be able to disentangle

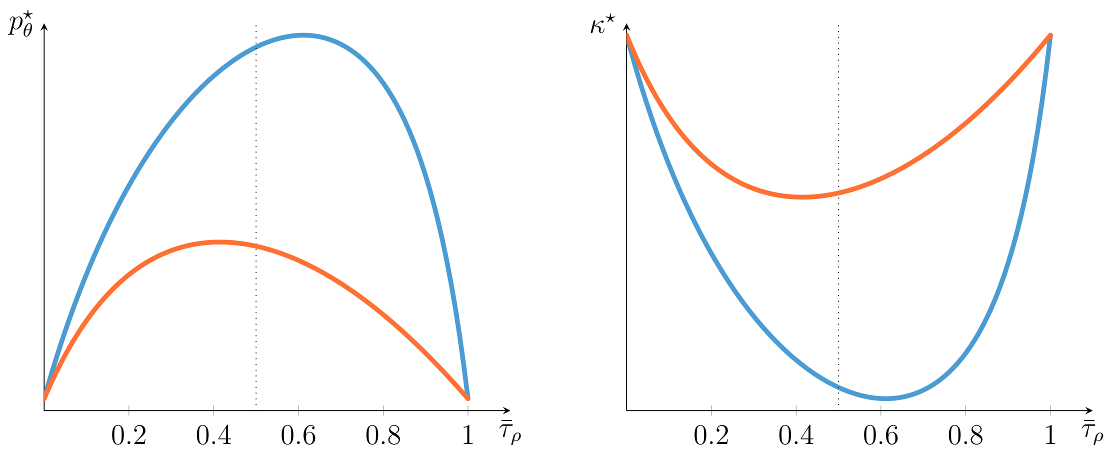

Figure 2: The firm's ability to maximize stage profits while confounding information is maximized when there is an intermediate amount of information regarding common costs ($\overline{\tau}_\rho$ interior). When there is either no or complete information regarding common costs ($\overline{\tau}_\rho \in \{0, 1\}$), private information regarding specific costs cannot be hidden, and cost-misrepresentation incentives are maximized. Where price responsiveness is maximized and information transmission is minimized depends on whether goods are complements (orange curve) or substitutes (blue curve). The ability of price to signal specific information ($\kappa^\star$) is inversely proportional to the sensitivity of price to information about specific costs.

the noise in the pricing strategy. Therefore, for a given value of $\bar{\tau}_\theta$, an intermediate level of precision $\bar{\tau}_\rho$ will maximize $p_\theta^\star$ and minimize informativeness $\kappa^\star$.

In general the incentive to hide specific cost information leads firms to be less responsive to their specific cost signal than is optimal in a one period game (without the informational channels implied by our two period model) or when firms sell independent products: $p_\theta^* < 1/2$ when $e \neq 0$. Firms will increase the sensitivity of first period prices to information about the specific cost component when incentives to signal jam are relaxed. Since the second dimension of uncertainty introduces noise into equilibrium pricing decisions, prices will be more responsive to information about specific costs than in a model without a common cost component.

3.2 Extension to large markets

To understand the impacts of this new inference channel on larger markets, we analyze the n-firm analogue of our basic model. In each period t, firm i's demand is

$$q_{itn}\left(p_{itn}, p_{-itn}\right) = \frac{1}{n-1}\left(a - bp_{itn} + \frac{e}{n-1}\sum_{j\neq i} p_{jtn}\right). \tag{1}$$

When $n = 2$ the normalization terms $n - 1$ are identically 1; this returns our benchmark model, $q_{it2} \equiv q_{it}$. All other assumptions from the benchmark model — for example, conditionally independent signals of ρ — are maintained.

We maintain our focus on symmetric equilibria in linear pricing strategies. Unlike the benchmark model we do not establish uniqueness. We show that the form of equilibrium pricing coefficients depends in a natural way on the number of firms. The linear equilibrium analysis of the n-firm extension is not substantially different from that of the base case, and we omit most of the basic calculations; details are found in Appendix A.

Theorem 2. *In the linear equilibrium of the n-firm model,*

$$p^\star_{i1n}\left(s_{i\theta}, s_{i\rho}\right) = p^\star_{0n} + p^\star_{\theta n}\mathbb{E}\left[\theta_i \middle| s_{i\theta}\right] + p^\star_{\rho n}\mathbb{E}\left[\rho \middle| s_{i\rho}\right],$$

where

$$p^\star_{\theta n} = \frac{1}{2 + \beta_n \kappa^\star_n}, \quad p^\star_{\rho n} = \frac{1 - \left(\frac{1-r}{2-r}\right)\beta_n \kappa^\star_n}{2 - r\overline{\tau}_\rho - \frac{1}{2}\left(1 - \overline{\tau}_\rho\right)\beta_n^2 {\kappa^\star_n}^2},$$

$$\kappa^\star_n = \frac{\sigma^2_\theta \bar{\tau}_\theta p^\star_{\theta n}}{\sigma^2_\rho\left(1 - \bar{\tau}_\rho\right)\bar{\tau}_\rho {p^\star_{\rho n}}^2 + \sigma^2_\theta \bar{\tau}_\theta {p^\star_{\theta n}}^2} \quad \textit{and} \quad \beta_n = \frac{r^2}{(2-r)\left(2(n-1) + r\right)}.$$

An indirect implication of Theorem 2 is that equilibrium inference from prices does not substantively change in the extension to n firms. That is, what firm $j \neq i$ can learn about firm i's specific cost θ_i from its first period price p_{i1n} depends only on $\kappa^\star_n$, which retains the same form as in the base case. It is not the case that $\kappa^\star$ is unchanged from the base case to the n-firm extension, but $\kappa^\star_n$ depends in the same way on $p^\star_{\theta n}$ and $p^\star_\rho$ regardless of the number of firms. Equivalently, $\kappa^\star$ and $\kappa^\star_n$ are identical functions of different price coefficients, $p^\star_\theta$ and $p^\star_\rho$ versus $p^\star_{\theta n}$ and $p^\star_{\rho n}$, respectively. Intuitively this is straightforward: once ρ is known, firm j can separate inferences about firm i's initial information from inferences about firm k's initial information. Because signals of ρ are conditionally uncorrelated, nothing learned about firm k can affect what is learned about firm i. If the independence assumption were relaxed, or if ρ were not made public in the second period, this would no longer be the case.

The linear equilibrium of the n-firm extension retains the cross-dependency of price coefficients and inference. Even when the number of firms is large, inference about any particular firm remains relatively stable. How information affects strategies when markets are large depends mostly on the incentive of any one firm to hide information from its opponents. As it turns out, in the limit no firm faces any incentive to obfuscate its private information. This is intuitive, as when the market is large any firm is one of many; since individual complementarities (or substitutabilities) $e/(n-1)$ are going to zero as the market becomes large, exposing private information does not dramatically affect opponent pricing incentives. This is true even though aggregate complementarities $\sum_{j\neq i} e/(n-1) = e$ are held constant.[28]

It is straightforward to see that equilibrium inference $\kappa_n^\star$ is bounded.[29] Then the effect of firm i's revelation, $\beta_n \kappa_n^\star$, goes to 0 as n becomes large, since

$$\lim_{n\nearrow\infty} \beta_n = \lim_{n\nearrow\infty} \frac{r^2}{(2-r)\left(2\left(n-1\right)+r\right)} = 0.$$

This implies a simple analytic form for equilibrium prices with a large number of firms.

Corollary 1. *In the linear equilibrium of the large-n extension, equilibrium prices are*

$$p_{i1\infty}^\star\left(s_{i\theta}, s_{i\rho}\right) = p_{0\infty}^\star + p_{\theta\infty}^\star \mathbb{E}\left[\theta_i \middle| s_{i\theta}\right] + p_{\rho\infty}^\star \mathbb{E}\left[\rho \middle| s_{i\rho}\right],$$

where

$$p_{0\infty}^\star = \frac{1}{2-r}\left(\frac{a}{b} + \frac{1}{2} r\mu_\theta + \mu_\rho\right) - \frac{\mu_\rho}{2 - r\overline{\tau}_\rho}, \quad p_{\theta\infty}^\star = \frac{1}{2}, \quad \textit{and} \;\; p_{\rho\infty}^\star = \frac{1}{2 - r\overline{\tau}_\rho}.$$

The lack of incentives to hide information is immediate in Corollary 1. $p_{\theta\infty}^\star = 1/2$ is exactly the dependence of price on private cost information in the equivalent monopoly problem. The value of $p_{\rho\infty}^\star$ is similar to the dependence of price on commonly known costs in a standard oligopoly problem. This dependence is adjusted by $\overline{\tau}_\rho$ to account for the fact that firm i's beliefs about firm j's beliefs are a reversion to the mean of firm i's ex ante beliefs. That is, if firm i believes $\mathbb{E}[\rho|s_{i\rho}] = \tilde{\rho}_i < \mu_\rho$, firm i believes that firm j believes $\mathbb{E}[\rho|s_j] \in (\tilde{\rho}_i, \mu_\rho)$.

4 Information sharing

We now consider the effect of firms sharing cost information — for example, as through a trade association — on expected surplus and pricing strategies. When information is shared, we assume that signals about the common cost component are verified and made public, while signals of the firms' specific costs are not revealed. Information about common costs is the unique information that is directly relevant to all firms' pricing decisions; for example, this can reflect information about the market for shared inputs. We first look at

[28] In the limit there is the additional question of whether the aggregation of these opponent incentives results in a strictly positive effect on the firm's incentives to hide information. Our results answer this in the negative.

[29] By definition, $\kappa_n^\star \leq 1/p_{\theta n}^\star$. Since $p_{\theta n}$ is bounded away from zero, $\kappa_n^\star$ is bounded above. A full proof is provided in Appendix D.

the impact of information sharing in the duopoly setting. Section 4.1 analyzes the impact on large markets; Section 5 considers more general information sharing agreements.[30]

When firms share their signals about the common cost component they will have the same information about this parameter, and therefore the same expectation of its value. While there are still two cost components, the informational structure is simplified so that firms only possess private information about their specific cost components; the remaining uncertainty regarding the common cost component is shared by both firms, as they have shared their signals regarding the common cost component ρ. While the optimality conditions look similar in this setting, equilibrium pricing strategies in the first period fully reveal the private information of each firm. We outline the significant differences from the previous section.

In the second period, the information that is available to each firm now includes both common cost signals, $s_\rho \equiv (s_{i\rho}, s_{j\rho})$. The new first order conditions are given in Lemma 5.

Lemma 5. *Firm i's optimal second period price is*

$$p_{i2}^c = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p_{j2}^c\middle|\,\rho, s_\rho, \mathbf{p_1}\right]\right).$$

Firm i's optimal first period price for a given pricing strategy of firm j is

$$\begin{aligned}\hat{p}_{i1} = &\left(\frac{1}{2b}\right)\mathbb{E}\left[bc_i + a + ep_{j1}\middle|\, s_\rho, s_{i\theta}\right] \\ &+ e\left(\frac{1}{2b}\right)^2\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}\middle|\,\rho, s_\rho, \hat{p}_{i1}, p_{j1}\right]\right)\frac{\partial}{\partial p_{i1}}\mathbb{E}\left[p_{j2}^c\middle|\,\rho, s_\rho, \hat{p}_{i1}, p_{j1}\right]\middle|\, s_\rho, s_{i\theta}\right].\end{aligned}$$

In a symmetric linear equilibrium, the first period price given signals $(s_{i\theta}, s_{i\rho})$ is $p_{i1}^c = p_0^c + p_\theta^c\mathbb{E}[\theta_i|s_{i\theta}] + p_\rho^c\mathbb{E}[\rho|s_\rho]$. Because s_ρ and p_{i1} are publicly observable, the value of $s_{i\theta}$ can be inferred by competitors. Therefore the expectation of each firm's cost in the second period, given publicly available information, is $\mathbb{E}[c_i|\rho, s_\rho, p_{i1}] = \rho + \mathbb{E}[\theta_i|s_{i\theta}]$, where $s_{i\theta}$ can be determined from the first period price p_{i1}. The impact of firm i's first period price on firm j's second period price is

$$\frac{\partial}{\partial p_{i1}}\mathbb{E}\left[p_{j2}^c\middle|\,\rho, p_{i1}^c\right] = \frac{r}{4-r^2}\kappa^c, \text{ where } \kappa^c \equiv \frac{\partial}{\partial p_{i1}}\mathbb{E}\left[c_i\middle|\,\rho, s_\rho, p_{i1}^c\right] = \frac{1}{p_\theta^c}.$$

Given the simplified information structure after information sharing, first period prices

[30] Throughout Section 4 we assume that only information about the common cost component may be shared. When firms share (verifiable) information about both cost components they initially have identical information about each others' costs, and there is no incentive to soften competition. When firms share information about their specific costs but *not* about the common cost signal, there is also no incentive to soften competition, since the common cost parameter ρ is learned before the second period. In either case, what remains is a more or less standard two period oligopoly analysis.

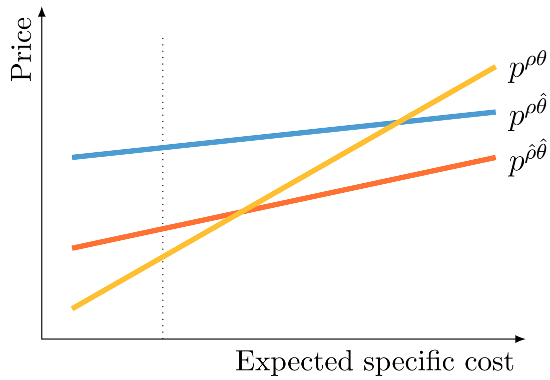

Figure 3: With private information about shared costs, prices ($p^{\hat{\rho}\hat{\theta}}$) are more responsive to information about specific costs than when there is no private information about shared costs ($p^{\rho\hat{\theta}}$), but less responsive to information than when all information is shared ($p^{\rho\theta}$). Expected prices (where conditional expected costs equal unconditional expected costs, the dotted line) are higher when firms share information, and $p^{\rho\hat{\theta}}$ will intersect $p^{\rho\theta}$ to the right of the intersection with $p^{\hat{\rho}\hat{\theta}}$.

are more informative about private signals of specific costs, and therefore firms have a stronger incentive to misrepresent costs. This leads firms to use less of thier specific cost information when choosing first period prices.

Proposition 3. *In the unique equilibrium in linear pricing strategies the coefficient on specific cost information is less than the corresponding coefficient in the equilibrium without information sharing:*

$$p^c_\theta = \frac{1-\beta}{2} \leq p^\star_\theta.$$

Additionally, prices are more informative than in the corresponding equilibrium without information sharing: $\kappa^c \geq \kappa^\star$. Both inequalities are strict when $e \neq 0$.

Because firms share common cost information, second period prices are more responsive to the first period prices (versus the setting without information sharing). In this setting, it is easier to soften future competition and therefore firms have a greater incentive to choose a higher first period price. The increase in expected price imposes a first order negative effect on consumer welfare; as we show later, this effect may be dominated by effects on the variance and covariance of prices.

Theorem 3. *Expected first period prices are higher when firms share signals about common cost information, $\mathbb{E}[p^\star_{i1}] \leq \mathbb{E}[p^c_{i1}]$ where the inequality is strict when $e \neq 0$. Moreover, expected second period prices are unaffected by information sharing.*

The first order conditions in Lemmas 3 and 5 appear identical, but have two distinctions. First, optimal first period prices depend on the opponent's expected second period price.

However, the difference in information structure does not change expected prices in the second period since optimal prices are linear in the beliefs about the competing firm's costs, and on average these beliefs must be correct. Second, first period prices depend on the rate at which an increase in first period price increases the competitor's second period price. From Proposition 3, the rate of increase is higher when firms share common cost information, thus expected first period prices are higher when information is shared.

An increase in expected prices will tend to increase producer surplus and decrease consumer surplus. However, surplus is also affected by the variance and covariance of prices. To fully consider the welfare effects of sharing common cost information, we specify the following utility which induces the given linear demand structure.

$$u(\mathbf{q};\mathbf{p}) = \frac{a}{b-e}(q_i+q_j) - \frac{1}{2}\left(\frac{b}{b^2-e^2}\right)(q_i^2+q_j^2) - \left(\frac{e}{b^2-e^2}\right)q_iq_j - (p_iq_i+p_jq_j) \tag{2}$$

From this specification, equilibrium expected consumer and producer surplus can be derived.

Lemma 6. *In a symmetric linear equilibrium, expected consumer surplus in each period of competition is*

$$\mathbb{E}\left[u\left(\mathbf{p}_\mathbf{t}^\star\right)\right] = \left(-2a+(b-e)\mathbb{E}\left[p_{it}^\star\right]\right)\mathbb{E}\left[p_{it}^\star\right] + b\operatorname{Var}\left(p_{it}^\star\right) - e\operatorname{Cov}\left(p_{it}^\star, p_{jt}^\star\right)$$

When expected demand is positive, it is the case that $a \geq (b-e)\mathbb{E}[p_i]$. Then the expression in Lemma 6 is decreasing in $\mathbb{E}[p_{it}^\star]$ and $e\operatorname{Cov}(p_{it}^\star, p_{jt}^\star)$ and increasing in $b\operatorname{Var}(p_{it}^\star)$. Higher average prices harm consumers, as does correlation in prices when goods are substitutes ($e > 0$). More volatile prices benefit consumers — expected surplus losses are dominated by expected surplus gains — as does correlation in prices when goods are complements ($e < 0$). That the effect of correlation depends on substitutability follows from the fact that correlation increases the variance of the average purchase price of a good within a bundle when goods are complements, and decreases this variance when goods are substitutes.

Lemma 7. *In a symmetric linear equilibrium, expected producer surplus in each period t of competition is given by*

$$\begin{aligned}\mathbb{E}\left[\Pi_t^\star\right] = 2\big[&\left(a-(b-e)\,\mathbb{E}\left[p_{it}^\star\right]\right)\left(\mathbb{E}\left[p_{it}^\star\right]-\mathbb{E}\left[c_i\right]\right) + b\left(\operatorname{Cov}\left(c_i,p_{it}^\star\right)-\operatorname{Var}\left(p_{it}^\star\right)\right)\\ &-e\left(\operatorname{Cov}\left(c_i,p_{jt}^\star\right)-\operatorname{Cov}\left(p_{it}^\star,p_{jt}^\star\right)\right)\big].\end{aligned} \tag{3}$$

In addition to the expectation, covariance and variance of equilibrium prices, producer surplus also depends on the correlation between equilibrium prices and realized cost. From Theorem 3, first period prices are higher when firms share industry cost information (and expected second period prices are unaffected). The price coefficients p_θ and p_ρ do not depend on the demand intercept a, so the variance and covariance of prices are independent of a. Then when demand is relatively inelastic (the parameter a is large relative to b and e) the change in expected price will dominate all other welfare effects from information sharing.[31] When goods are substitutes, higher expected prices increase profits, all else equal. When goods are complements, there are competing effects, since a firm's demand drops as its opponent's price increases. Proposition 4 shows that producer surplus increases (decreases) with an information sharing agreement when goods are substitutes (complements) and demand is sufficiently inelastic. The welfare impacts of an information sharing agreement are illustrated in Figure 4, and are later summarized in Table 1.

Proposition 4. *Sharing common cost information decreases consumer surplus when goods are sufficiently substitutable ($e \approx b$), or when goods are related ($e \neq 0$) and demand is sufficiently inelastic ($a \gg b$). Sharing common cost information increases producer surplus when goods are substitutes ($e > 0$) and decreases producer surplus when goods are complements, provided demand is sufficiently inelastic ($a \gg b$).*

Remark 2. *Although we model Bertrand competition, our analytical approach can be straightforwardly applied to Cournot competition. As is familiar from the literature (see, e.g., Vives [1984]), the basic structure of equilibrium is unaffected by the mode of competition, but the sign of welfare effects is reversed. For example, under Cournot competition, sharing common cost information* increases *expected consumer surplus when demand is inelastic ($a \gg b$).*

Remark 3. *In our analysis we take the precision of information to be exogenous. In reality, firms may allocate resources to improve their estimates of cost parameters. Consider the effect of an (unobserved) marginal increase in precision of cost information on the firm's profits over the two periods of competition. By the envelope theorem, a marginal increase in precision affects first period profits only through the ex ante variance and covariance of prices, and not through price selection. The incentive to signal jam when sharing common cost information reduces the extent to which firms use information about specific costs (Proposition 3). Therefore, information sharing reduces marginal incentives to acquire information regarding specific costs.*

[31] Because expected second period prices are unaffected by information sharing, the effect of information sharing on second period consumer and producer surplus is determined by the effect on the variance of prices, and the covariance of prices and costs. The magnitude of these effects is independent of demand elasticity, and therefore when demand is very inelastic the effect of increased first period price dominates all other considerations.

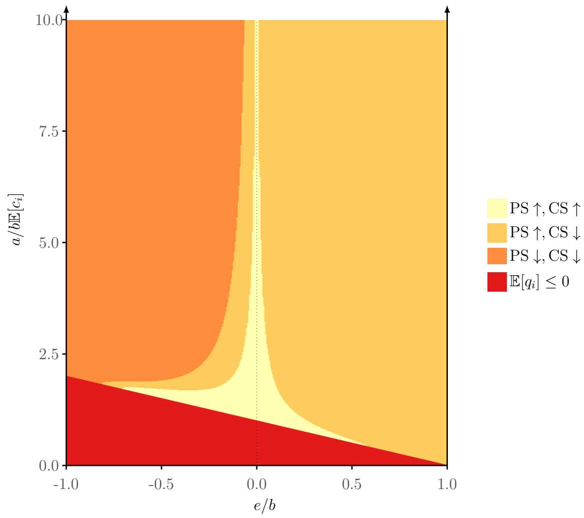

Figure 4: The surplus effects of information sharing (Proposition 4). When demand is inelastic ($a \gg b$), information sharing increases producer surplus when goods are substitutes and decreases producer surplus when goods are complements. Information sharing decreases consumer surplus when demand is inelastic, provided goods are related ($e \neq 0$), and when goods are very substitutable ($e \approx b$). When $e \approx 0$, surplus is approximately unaffected by information sharing, and second-order effects dominate the comparison.

4.1 Information sharing in large markets

Proposition 4 shows that expected producer surplus increases and expected consumer surplus falls when information is shared, provided $a \gg b \geq e > 0$. A more general comparison is hampered by the size of the parameter space: comparisons of welfare across regimes will in general depend not only on the demand specification (as illustrated in Figure 4) but also on the information structure induced by noisy cost signals. As noted in Proposition 2, equilibrium price coefficients and inference are not monotone in precision $\overline{\tau}_\rho$, making it difficult to directly apply standard methods from comparative statics.

These comparisons are simplified when the market is large. In the linear equilibrium with a large number of firms, the expectation of first period prices is independent of $\overline{\tau}_\rho$. While $p^\star_{\rho\infty}$ depends on $\overline{\tau}_\rho$, in expectation this is exactly offset by the $\mu_\rho p^\star_{\rho\infty}$ term in $p^\star_{0\infty}$. Then to the extent that information sharing (an increase in $\overline{\tau}_\rho$) alters producer or consumer surplus, it is through the coefficients $p^\star_{0\infty}$, $p^\star_{\rho\infty}$ appearing in the variance and covariance of first period prices. Second period strategies, and therefore second period surplus, are unaffected by information sharing.[32]

[32]Common costs ρ are commonly known in the second period, allowing potential inference of opponent information from first period prices. However the pricing strategies are symmetric, linear and depend only

Lemma 8. *There exist constants $C_u, C_\pi \in \mathbb{R}$ such that for any $\bar{\tau}_\rho$, first period consumer and producer surplus in a linear equilibrium of the large-n extension are given by*

$$\begin{aligned}
\mathbb{E}\left[u_{1\infty}\right] &\propto (1-r)\operatorname{Var}\left(p^\star_{i1\infty}\right) - r\operatorname{Cov}\left(p^\star_{i1\infty}, p^\star_{j1\infty}\right) + C_u,\\
\mathbb{E}\left[\Pi_{1\infty}\right] &\propto \left(\operatorname{Cov}\left(c_i, p^\star_{i1\infty}\right) - \operatorname{Var}\left(p^\star_{i1\infty}\right)\right) - r\left(\operatorname{Cov}\left(c_i, p^\star_{j1\infty}\right) - \operatorname{Cov}\left(p^\star_{i1\infty}, p^\star_{j1\infty}\right)\right) + C_\pi,
\end{aligned}$$

where i, j are any firms such that $i \neq j$.

When information is shared in the first period, aggregation of an infinite number of informative signals is equivalent to common knowledge of ρ prior to setting first period prices.[33] Firms will then choose prices as in Corollary 1 where $\mathbb{E}[\rho|s_\rho] = \rho$ and $\bar{\tau}_\rho = 1$. Propositions 5 and 6 summarize the welfare impact of sharing information when the number of firms is large. Information sharing never harms producer surplus and in almost all cases strictly increases it.

Proposition 5. *When the number of firms is large, information sharing strictly increases producer surplus.*

The impact of information sharing on consumer surplus depends on $\bar{\tau}_\rho$ prior to information sharing and substitutability of the firm's products r. Specifically, when $r \leq (\sqrt{33}-5)/2 \approx 0.372$, i.e. products are complements or weakly substitutable, information sharing always improves consumer surplus. For intermediate values of r, surplus increases when information is relatively dispersed prior to sharing, $\bar{\tau}_\rho \ll 1$. For low initial precision the ability to tie prices more directly to costs outweighs strategic effects, while for high initial precision there is not much information gained when signals are shared, and strategic effects dominate. When goods are relatively substitutable, $r \geq 1/2$, consumer surplus will be harmed for any initial value of $\bar{\tau}_\rho$.

Proposition 6. *When goods are complements, consumer surplus is increasing in precision $\bar{\tau}_\rho$. When goods are substitutes information sharing increases consumer surplus when $r < 1/2$ and $\bar{\tau}_\rho \ll 1$ prior to sharing, and decreases consumer surplus when $r \geq 1/2$.*

on expected opponent prices. Then an optimal strategy reduces to a linear strategy on expected opponent costs. With a large number of firms the law of large numbers applies, and the sum of expected opponent costs is equivalent to an average opponent cost. This is independent of whether or not information is shared.

[33]This is the case when $\bar{\tau}_\rho > 0$. When $\bar{\tau}_\rho = 0$ aggregation yields no additional information, but this problem remains equivalent to profit maximization with only firm specific costs.

5 Generalized information sharing

In Section 4, we considered an all-or-nothing information sharing agreement. Although firms share information about common costs and not about specific costs, firms either share all information about a particular signal, or no information about that signal. In this section we generalize the analysis to consider partial information sharing agreements. Specifically, firms may share information which contains an arbitrary portion of each of their two signals. As an example, consider the aggregation of information about transportation costs. This might represent half of all costs that are common to the firms and twenty percent of costs that are individual to each firm. We maintain the assumption that both initial information and the information shared are symmetric in type, amount and precision.

Formally, we assume that firm i receives a set of signals regarding cost x, $s_{ixm} = x + \varepsilon_{ixm}$, where $x \in \{\rho, \theta_i\}$, and $m \in \{1, \ldots, M_x\}$. Each error term ε_{ixm} is normally distributed with source-dependent variance, $\varepsilon_{ixm} \sim N(0, \sigma^2_{\varepsilon x})$, and is independent of $\varepsilon_{i'x'm'}$ for $(i, x, m) \neq (i', x', m')$. Then, prior to potential infomation sharing, firm i's signal about cost x is $s_{ix} = x + \sum_{m=1}^{M_x} \varepsilon_{ixm}/M_x$.[34]

Firms abide by an information sharing agreement which determines which signals s_{ixm} to share with their competitors. Because ε_{ixm} is independent of $\varepsilon_{ixm'}$, it is sufficient to consider only the number of signals shared, and not the specific identities. Let $\tilde{M}_{i \to jx}$ be the number of firm i's signals about cost x which are shared with firm j.[35] Because the ε_{ixm} terms are independently and identically distributed, the effects of information sharing agreements depend only on the number of signals shared, and not on which signals are shared (provided the set of shared signals is an ex ante commitment); without loss of generality, we therefore assume that firm j shares signals s_{jxm} for $m \in \{1, \ldots, \tilde{M}_{j \to ix}\}$. Consistent with the definition of $\tilde{M}_{i \to jx}$, we decorate post-sharing variables with tildes. After information sharing, firm i's signal about cost $x \in \{\rho, \theta_i, \theta_j\}$ is $\tilde{s}_{ix}$, where

$$\tilde{s}_{i\rho} = \rho + \frac{1}{M_\rho + \tilde{M}_{j \to i\rho}} \left(\sum_{m=1}^{M_\rho} \varepsilon_{i\rho m} + \sum_{m=1}^{\tilde{M}_{j \to i\rho}} \varepsilon_{j\rho m} \right) = \frac{1}{M_\rho + \tilde{M}_{j \to i\rho}} \left(\sum_{m=1}^{M_\rho} s_{i\rho m} + \sum_{m=1}^{\tilde{M}_{j \to i\rho}} s_{j\rho m} \right),$$

$$\tilde{s}_{i\theta_i} = \theta_i + \frac{1}{M_\theta} \sum_{m=1}^{M_\theta} \varepsilon_{i\theta m} = \frac{1}{M_\theta} \sum_{m=1}^{M_\theta} s_{i\theta m}, \quad \tilde{s}_{i\theta_j} = \theta_j + \frac{1}{\tilde{M}_{j \to i\theta}} \sum_{m=1}^{\tilde{M}_{j \to i\theta}} \varepsilon_{j\theta m} = \frac{1}{\tilde{M}_{j \to i\theta}} \sum_{m=1}^{\tilde{M}_{j \to i\theta}} s_{j\theta m}.$$

[34] The variance of this signal is $\sigma^2_{\varepsilon x}/M_x$. This can be mapped to our base model by rescaling the variance $\sigma^2_{\varepsilon x}$ by M_x.

[35] This notation is related to that presented in Vives [2001]. In the appendix, we shorten equations by writing $\tilde{M}_{jx} \equiv \tilde{M}_{i \to jx}$, but for clarity of exposition we retain the "giving to" notation here in the main text.

The precision of information on each component after sharing is

$$\tilde{\tau}_{i\rho} = \frac{M_\rho + \tilde{M}_{j\to i\rho}}{\sigma^2_{\varepsilon\rho}}, \quad \tilde{\tau}_{i\theta_i} = \frac{M_\theta}{\sigma^2_{\varepsilon\theta}}, \text{ and } \tilde{\tau}_{i\theta_j} = \frac{\tilde{M}_{j\to i\theta}}{\sigma^2_{\varepsilon\theta}}.$$

Following previous notation, we define $\tilde{\overline{\tau}}_{ix} = \tilde{\tau}_{ix}/(\tau_x + \tilde{\tau}_{ix})$.

Note that, unlike in our base model, firms may directly share information about their specific costs. Then firm i has a signal $\tilde{s}_{i\theta_j}$ regarding firm j's specific costs, and equilibrium prices will typically respond to this information. Firm i's first period price $\tilde{p}_{i1}$ will depend not only on its signal of firm j's specific costs, $\tilde{s}_{i\theta_j}$, but also on what it knows to be shared information between the two firms about its own specific costs, $\tilde{s}_{j\theta_i}$ and common costs $\tilde{s}_\rho$. The definition of an equilibrium pricing strategy from Section 2 must be relaxed to account for these additional signals, and allow $\tilde{p}_{i\theta_j} \neq 0$.

Given the firms engage in information sharing, the history for firm i in period 1 is $h_{i1} = (\tilde{s}_{i\rho}, \tilde{s}_{i\theta_i}, \tilde{s}_{i\theta_j}, \tilde{s}_{j\theta_i}, \tilde{s}_\rho)$. In a linear strategy, the first period price for each history is

$$\tilde{p}_{i1}(h_{i1}) = \tilde{p}_0 + \tilde{p}_{i\rho}\mathbb{E}\left[\rho \left|\tilde{s}_{i\rho}\right.\right] + \tilde{p}_{i\tilde{s}_\rho}\mathbb{E}\left[\rho\left|\tilde{s}_\rho\right.\right] + \tilde{p}_{i\theta_i}\mathbb{E}\left[\theta_i\left|\tilde{s}_{i\theta_i}\right.\right] + \tilde{p}_{i\tilde{s}_{j\theta_i}}\mathbb{E}\left[\theta_i\left|\tilde{s}_{j\theta_i}\right.\right] + \tilde{p}_{i\theta_j}\mathbb{E}\left[\theta_j\left|\tilde{s}_{i\theta_j}\right.\right].$$ [36]

Second period prices and profits are as in Lemma 2, where public information is given by $\tilde{h}_2 = (\rho, \mathbf{p_1}, \tilde{s_\rho}, \tilde{s}_{j\theta_i}, \tilde{s}_{i\theta_j})$. First period prices are characterized in Theorem 4.

Theorem 4. *In the linear equilibrium under a generalized information sharing agreement, first period price coefficients are*

$$\tilde{p}_{i\theta_i} = \tfrac{1}{2+\beta\tilde{\kappa}}, \quad \tilde{p}_{i\theta_j} = \tfrac{1}{2}\left(\tfrac{r}{4-r^2}\right)(2-\beta\tilde{\kappa}) + \tfrac{1}{2r}\left(\tfrac{4+r^2}{4-r^2}\right)\beta^2\tilde{\kappa},$$
$$\tilde{p}_{i\rho} = \frac{1-\left(\frac{1-r}{2-r}\right)\beta\tilde{\kappa}}{2-(1-\eta_{i\rho})\left(r\tilde{\overline{\tau}}_{j\rho}+\frac{1}{2}\beta^2\tilde{\kappa}^2(1-\tilde{\overline{\tau}}_{j\rho})\right)},$$
$$\tilde{p}_{i\tilde{s}_{j\theta_i}} = \tfrac{1}{2}r\tilde{p}_{i\theta_j} + \tfrac{1}{4}\beta^2\tilde{\kappa}(1-\tilde{\kappa}\tilde{p}_{i\theta_i}), \quad \textit{and} \quad \tilde{p}_{i\tilde{s}_\rho} = \tfrac{1}{2}\left(\tfrac{2r-\beta^2\tilde{\kappa}^2}{2-r}\right)\eta_{i\rho}\tilde{p}_{i\rho}$$

where

$$\eta_{i\rho} = \frac{2\tilde{M}_{j\to i\rho}}{M_\rho + \tilde{M}_{j\to i\rho}} \quad \textit{and}$$
$$\tilde{\kappa} = \frac{\left(1-\frac{\sigma_\theta^2+\left[\sigma^2_{\varepsilon\theta}/M_\theta\right]}{\sigma_\theta^2+\left[\sigma^2_{\varepsilon\theta}/\tilde{M}_{j\to i\theta}\right]}\right)\tilde{p}_{j\theta_j}\tilde{\overline{\tau}}_{j\theta_j}\sigma^2_\theta}{\left(1-\frac{\sigma_\theta^2+\left[\sigma^2_{\varepsilon\theta}/M_\theta\right]}{\sigma_\theta^2+\left[\sigma^2_{\varepsilon\theta}/\tilde{M}_{j\to i\theta}\right]}\right)\tilde{p}^2_{j\theta_j}\tilde{\overline{\tau}}_{j\theta_j}\sigma^2_\theta + \left(1-\frac{\tilde{M}_{i\to j\rho}+\tilde{M}_{j\to i\rho}}{M_\rho+\tilde{M}_{i\to j\rho}}\right)\tilde{p}^2_{j\rho}\tilde{\overline{\tau}}_{j\rho}\left(1-\tilde{\overline{\tau}}_{j\rho}\right)\sigma^2_\rho}.$$

[36]This expression differs in appearance from the definition of a linear equilibrium in Section 2. This form allows clear expression of how different signal sources affect equilibrium prices. Because all expectations are linear in signal, these price coefficients may be aggregated to obtain the form in the definition in Section 2.

Remark 4. *Our benchmark model, with no information sharing, corresponds to* $\tilde{M}_{i\to j\theta} = \tilde{M}_{i\to j\rho} = 0$*. Our all-or-nothing information sharing model corresponds to* $\tilde{M}_{i\to jx} \in \{0, M_x\}$*. In both cases, substituting in for* $\tilde{M}_{i\to jx}$ *in Theorem 4 returns the equilibrium price coefficients from our earlier analyses. Note that if no information is shared about, e.g., specific cost* θ_i*, the conditional expectation* $\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}] = \mu_\theta$ *is independent of the signals* $r_{ix\theta}$*, and the coefficients* $\tilde{p}_{i\theta_j}$ *and* $\tilde{p}_{i\tilde{s}_{j\theta_i}}$ *are subsumed into* $\tilde{p}_0$*.*

Corollary 2. *Let* $\tilde{M}_{j\to ix} = \tilde{M}_{i\to jx} = \tilde{M}_x$*, and define* $\lambda_x = \tilde{M}_x / M_x$*. Suppose that the variance of error term* ε_{ixm} *is rescaled to match the base model,* $\mathrm{Var}(\varepsilon_{ixm}) = M_x\sigma^2_{\varepsilon x}$*. In the symmetric linear pricing equilibrium with generalized information sharing, the informational parameters are given by*

$$\eta_\rho = \frac{2\lambda_\rho}{1+\lambda_\rho}, \ \ \textit{and}\ \tilde{\kappa} = \frac{\left(1 - \frac{\sigma^2_\theta + \sigma^2_{\varepsilon\theta}}{\sigma^2_\theta + \left[\sigma^2_{\varepsilon\theta}/\lambda_\theta\right]}\right)\tilde{p}_{j\theta_j}\tilde{\tilde{\tau}}_{j\theta_j}\sigma^2_\theta}{\left(1 - \frac{\sigma^2_\theta + \sigma^2_{\varepsilon\theta}}{\sigma^2_\theta + \left[\sigma^2_{\varepsilon\theta}/\lambda_\theta\right]}\right)\tilde{p}^2_{j\theta_j}\tilde{\tilde{\tau}}_{j\theta_j}\sigma^2_\theta + (1-\eta_\rho)\,\tilde{p}^2_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\left(1 - \tilde{\tilde{\tau}}_{j\rho}\right)\sigma^2_\rho}.$$

Compared to our model with all-or-nothing information sharing, the coefficient on θ_i does not change except through the change in the informativeness of the price, $\kappa \mapsto \tilde{\kappa}$. Therefore this coefficient and $\tilde{\kappa}$ have an inverse relationship (as in Proposition 2). Corollary 2 makes clear that when no information about specific costs is shared, $\lambda_\theta = 0$, price informativeness $\tilde{\kappa}$ is a version of our informativeness parameter κ when common-cost information may be fractionally shared; in this case, $\lambda_\rho = 0$ implies $\tilde{\kappa} = \kappa^\star$ and $\lambda_\rho = 1$ implies $\tilde{\kappa} = \kappa^c$.

Relative to the equilibrium in Section 3, the new pricing coefficients $\tilde{p}_{i\theta_j}$ and $\tilde{p}_{i\tilde{s}_{j\theta_i}}$ do not affect the informativeness of first period prices. Firm i knows the signal which is shared with firm j, and therefore any variance in firm j's price due to the shared signal is fully accounted for by firm i. It follows that these coefficients do not appear in $\tilde{\kappa}$; the effect of information sharing is fully captured in the reduction of variance of the conditional expectation of opponent specific cost.

We can now generalize the results of Section 4 about the impact of information sharing agreements on expected prices.

Proposition 7. *For a fixed information structure, an information sharing agreement which increases (decreases)* $\tilde{\kappa}$ *will cause ex-ante expected first period price to increase (decrease) and will not impact ex-ante expected second period prices.*

Given the level of expected prices can be ranked by the equilibrium informativeness of the prices, then the impact of an information sharing agreement on expected prices is determined by its impact on $\tilde{\kappa}$. When informativeness increases with sharing, then the agreement leads to higher prices, and vice versa. Specifically, ex ante expected prices are

$$\mathbb{E}[\tilde{p}_{i1}] = \frac{a + b\mathbb{E}[c_i]}{2b - e} + \frac{b(a - (b - e)\mathbb{E}[c_i])}{(2b - e)^2}\beta\tilde{\kappa} \quad \text{and} \quad \mathbb{E}[\tilde{p}_{i2}] = \frac{a + b\mathbb{E}[c_i]}{2b - e}.$$

When more information about common cost component is shared, prices become more informative about the firm's remaining private information, increasing its incentive to soften competition. On the other hand, when information about specific costs is shared price becomes less informative about the remaining private information, reducing the incentive to soften competition.

Proposition 8. *For a fixed information structure, sharing more information on the common cost component (larger $\tilde{M}_{j\to i\rho}$ for given M_ρ, or larger $\lambda_{i\rho}$) will increase the equilibrium informativeness of prices while sharing more information on the specific cost component (larger $\tilde{M}_{i\to j\theta}$ for given M_θ, or larger $\lambda_{i\theta}$) will decrease the informativeness of prices.*

Propositions 7 and 8 together imply that an information sharing agreement regarding only common cost information will increase expected prices, while an agreement to share information about specific costs will decrease expected prices.

Corollary 3. *When demand is sufficiently inelastic, $a \gg b$, sharing more information about the common cost component (specific cost component) decreases (increases) consumer surplus when goods are related ($e \neq 0$), and increases (decreases) producer surplus when goods are substitutes ($e > 0$) and decreases (increases) producer surplus when goods are complements ($e < 0$).*

Our earlier results on the surplus effects of information sharing (Proposition 4) relied on the relationship $\kappa^c \geq \kappa^\star$, and not on the specific values of κ^c and $\kappa^\star$. Then a version of Proposition 4 holds whenever $\tilde{\kappa}' > \tilde{\kappa}$, and its opposite holds whenever $\tilde{\kappa}' < \tilde{\kappa}$. Then Proposition 8 is sufficient to imply that consumer surplus is harmed when information is shared about the common cost component, and improved when information is shared about the specific cost components.

6 Conclusion

Firms in an industry typically have heterogeneous costs of production, but these costs may include a common component. When firms have idiosyncratic information about common costs, the precision of estimated costs may be improved by sharing information through a trade association. The competitive impact of sharing information about industry-wide costs depends on how information is used and inferred without an information sharing agreement. To this end, we study a dynamic pricing competition model that allows for uncertainty in common cost and specific cost parameters. We characterize the symmetric linear equilibrium of this model and use it to examine how information sharing affects competition and welfare within the industry.

In a setting with two firms, information sharing increases incentives for firms to soften competition, leading to higher average prices. In settings where demand is relatively inelastic or products are close substitutes, information sharing reduces consumer surplus while increasing producer surplus. As the number of firms in the market increases, the effect of competition softening is reduced; in particular, as the number of firms in the market becomes arbitrarily large no individual firm's price decision conveys useful additional information, and strategic signaling vanishes. In a market with a large number of firms, information sharing no longer has an impact on expected prices and can lead to both higher producer surplus and consumer surplus when products are not close substitutes and industry relevant information is dispersed among the firms.

Because sharing information about industry relevant costs may lead to higher producer surplus, agreements to share this information may not stem from purely collusive motives. In fact, there are cases where an information sharing agreement between many firms can improve both producer and consumer surplus. However, our results suggest increased consumer surplus is less likely for firms that sell products that are close substitutes. In particular, these agreements can be a concern for competition in concentrated markets where the agreements can lead to higher and more coordinated prices even in the absence of an explicit or implicit collusive agreement.

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A Proofs for Section 3

Proof of Lemma 1. This follows directly from firm i's first-order condition with respect to second period price,

$$
\begin{aligned}
&(a - bp_{i2}^{\star}) + \int exdF^{j}\left(x; \rho, \mathbf{p}_1\right)dx - \left(p_{i2}^{\star} - c_i\right)b = 0 \\
\implies \quad & p_{i2}^{\star} = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[\left. p_{j2}^{\star}\right| \rho, \mathbf{p}_1\right]\right).
\end{aligned}
$$

Substituting in to the firm's profit function yields the expression in Lemma 1. □

Proof of Lemma 2. This follows from Lemmas 11 and 12, derived for the model with n firms. We give a proof for the two-firm case below.

Because firm i's optimal second period price, given in Lemma 1, holds given any information set, it also holds in expectation. That is,

$$\mathbb{E}\left[p_{i2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]=\frac{1}{2b}\mathbb{E}\left[a+bc_i+ep_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right].$$

Then

$$2b\mathbb{E}\left[p_{i2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]-e\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]=a+b\mathbb{E}\left[c_i\middle|\,\rho,\mathbf{p}_1\right].$$

This gives two equations, one each for firm i and firm j, in two unknowns, $\mathbb{E}[p_{i2}^{\star}|\rho,\mathbf{p}_1]$ and $\mathbb{E}[p_{j2}^{\star}|\rho,\mathbf{p}_1]$. Algebraic rearrangement yields the first expression in Lemma 2, and substituting into the firm's profit function yields the second. □

Proof of Lemma 3. This follows from standard monopoly profit maximization and application Lemma 1 to firm i's second period profits,

$$\begin{aligned}&\max_p \mathbb{E}\left[\pi_{i1}\middle|\,s_{i\rho},s_{i\theta}\right]+\mathbb{E}\left[\pi_{i2}^{\star}\middle|\,s_{i\rho},s_{i\theta}\right]\\&=\max_p \mathbb{E}\left[\left(a-bp+ep_{j1}\right)\left(p-c_i\right)+\frac{1}{4b}\left(a-bc_i+e\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]\right)^2\middle|\,s_{i\rho},s_{i\theta}\right].\end{aligned}$$

Note that second period profits depend on p_{i1} only through $\mathbf{p}_1$'s effect on firm j's beliefs. Without substituting in with the expression in Lemma 1 we could have obtained a similar reduction by applying the envelope theorem (firm i's second period price is optimal, conditional on its first period price). Firm i's first order condition is

$$0=\mathbb{E}\left[\left(a+bc_i+ep_{j1}\right)-2b\hat{p}_{i1}+\frac{e}{2b}\left(a-bc_i+e\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]\right)\frac{\partial}{\partial p_{i1}}\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]\middle|\,s_{i\rho},s_{i\theta}\right].$$

Rearrangement gives the desired result. □

Lemma 9. *Expected costs conditional on second period information are*

$$\mathbb{E}\left[c_i\middle|\,\rho,\mathbf{p}_1\right]=\left(\mu_\theta+\mu_\rho\right)+\left(1-\kappa_i\overline{\tau}_\rho p_{i\rho}\right)\left(\rho-\mu_\rho\right)+\kappa_i\left(p_{i1}-\left(p_{i0}+p_{i\theta}\mu_\theta+p_{i\rho}\mu_\rho\right)\right),$$

$$\textit{subject to } \kappa_i=\frac{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}}{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2+\left(1-\overline{\tau}_\rho\right)\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2}.$$

Proof. Note that, conditional on ρ, p_{j1} conveys no information about c_i. Then consider the joint distribution of c_i, p_{i1}, and ρ. Let $\tau_x\equiv 1/\sigma_x^2$ be the precision of the random variable x, and let $\overline{\tau}_x\equiv\tau_{\varepsilon_x}/(\tau_x+\tau_{\varepsilon_x})$ be the relative precision of the signal s_x, $x\in\{\theta_i,\theta_j,\rho\}$. Under a linear pricing strategy,

$$\begin{aligned} p_{i1} &= p_{i0} + p_{i\theta}\mathbb{E}\left[\theta_i\middle|\, s_{i\theta}\right] + p_{i\rho}\mathbb{E}\left[\rho\middle|\, s_{i\rho}\right] \\ &= \left(p_{i0} + \left(1-\overline{\tau}_\theta\right)p_{i\theta}\mu_\theta + \left(1-\overline{\tau}_\rho\right)p_{i\rho}\mu_\rho\right) + p_{i\theta}\overline{\tau}_\theta s_{i\theta} + p_{i\rho}\overline{\tau}_\rho s_{i\rho}. \end{aligned}$$

Then c_i, ρ, and p_{i1} are jointly normal,

$$(c_i, \rho, p_{i1})^T \sim N\left(\begin{pmatrix} \mu_\theta + \mu_\rho \\ \mu_\rho \\ \mathbb{E}\left[p_{i1}\right] \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 + \sigma_\rho^2 & \sigma_\rho^2 & \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta} + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} \\ \sigma_\rho^2 & \sigma_\rho^2 & \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} \\ \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta} + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} & \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} & \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2 \end{pmatrix}\right).$$

Then the conditional expectation of c_i, given ρ and p_{i1}, is

$$\begin{aligned} &\mathbb{E}\left[c_i\middle|\, \rho, p_{i1}\right] = \left(\mu_\theta + \mu_\rho\right) + \Sigma_{12}\Sigma_{22}^{-1}\left(\left(\rho, p_{i1}\right)^T - \left(\mu_\rho, \mathbb{E}\left[p_{i1}\right]\right)^T\right), \\ &\Sigma_{12} = \left(\sigma_\rho^2, \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta} + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}\right), \quad \Sigma_{22} = \begin{pmatrix} \sigma_\rho^2 & \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} \\ \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} & \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2 \end{pmatrix}. \end{aligned}$$

Write the matrix product as $\Sigma_{12}\Sigma_{22}^{-1} = (m_{i1}, m_{i2})$. Then

$$\begin{aligned} m_{i1} &= \frac{1}{\left(\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\right)\sigma_\rho^2 - \overline{\tau}_\rho^2\sigma_\rho^4 p_{i\rho}^2}\left(\overline{\tau}_\theta\sigma_\theta^2\sigma_\rho^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^4 p_{i\rho}^2 - \overline{\tau}_\theta\overline{\tau}_\rho\sigma_\theta^2\sigma_\rho^2 p_{i\theta}p_{i\rho} - \overline{\tau}_\rho^2\sigma_\rho^4 p_{i\rho}^2\right) \\ &= \frac{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2 - \overline{\tau}_\theta\overline{\tau}_\rho\sigma_\theta^2 p_{i\theta}p_{i\rho} - \overline{\tau}_\rho^2\sigma_\rho^2 p_{i\rho}^2}{\left(\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\right) - \overline{\tau}_\rho^2\sigma_\rho^2 p_{i\rho}^2} = 1 - \kappa_i\overline{\tau}_\rho p_{i\rho}; \\ m_{i2} &= \frac{1}{\left(\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\right)\sigma_\rho^2 - \overline{\tau}_\rho^2\sigma_\rho^4 p_{i\rho}^2}\left(-\overline{\tau}_\rho\sigma_\rho^4 p_{i\rho} + \overline{\tau}_\theta\sigma_\theta^2\sigma_\rho^2 p_{i\theta} + \overline{\tau}_\rho\sigma_\rho^4 p_{i\rho}\right) \\ &= \frac{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}}{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \left(1-\overline{\tau}_\rho\right)\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2} = \kappa_i. \end{aligned}$$

The result is then immediate. □

Proof of Lemma 4. This follows immediately from Lemma 9. □

Lemma 10. *There is an equilibrium in symmetric linear pricing strategies, where*

$$\begin{aligned} &\kappa^\star = \frac{\sigma_\theta^2\overline{\tau}_\theta p_\theta^\star}{\left(1-\overline{\tau}_\rho\right)\overline{\tau}_\rho\sigma_\rho^2 p_\rho^{\star 2} + \overline{\tau}_\theta\sigma_\theta^2 p_\theta^{\star 2}}, \\ &\textit{subject to } p_\theta^\star = \frac{1}{2+\beta\kappa} \textit{ and } p_\rho = \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa^\star}{2 - r\overline{\tau}_\rho - \frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^{\star 2}}, \end{aligned}$$

where $r = e/b$ and $\beta = r^2/(4-r^2)$.

Proof. From Lemmas 3 and 4, first period prices are given by

$$4bp_{i1}^{\star} = 2\mathbb{E}\left[bc_i + a + ep_{j1}^{\star} \middle|\, s_{i\rho}, s_{i\theta}\right] + \mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}^{\star} \middle|\, \rho, p_{i1}^{\star}, p_{j1}^{\star}\right]\right) \beta\kappa_i \middle|\, s_{i\rho}, s_{i\theta}\right].$$

Lemma 2 gives second period expected prices,

$$\mathbb{E}\left[p_{j2}^{\star} \middle|\, \rho, \mathbf{p}_1\right] = \frac{1}{4b^2 - e^2}\left((2b+e)\, a + 2b^2 \mathbb{E}\left[c_j | \, \rho, p_{j1}\right] + be\mathbb{E}\left[c_i |\, \rho, p_{i1}\right]\right).$$

Following Lemma 9,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[c_j |\, \rho, \mathbf{p}_1\right] |\, s_{i\theta}, s_{i\rho}\right] &= \mu_\theta + \mathbb{E}\left[\rho |\, s_{i\rho}\right], \\
\mathbb{E}\left[\mathbb{E}\left[c_i |\, \rho, \mathbf{p}_1\right] |\, s_{i\theta}, s_{i\rho}\right] &= \mu_\theta + \mathbb{E}\left[\rho |\, s_{i\rho}\right] \\
&\qquad + \kappa_i p_{i\theta}\left(\mathbb{E}\left[\theta_i |\, s_i\right] - \mu_\theta\right) + \kappa_i p_{i\rho}\left(1 - \overline{\tau}_\rho\right)\left(\mathbb{E}\left[\rho |\, s_{i\rho}\right] - \mu_\rho\right).
\end{aligned}$$

Substituting in gives

$$\begin{aligned}
4bp_{i1}^{\star} &= 2\mathbb{E}\left[bc_i + a + ep_{j1}^{\star} \middle|\, s_{i\rho}, s_{i\theta}\right] + \beta\kappa_i \mathbb{E}\left[\left(a - bc_i + \frac{e}{4b^2 - e^2}\left((2b+e)\, a\right)\right) \middle|\, s_{i\rho}, s_{i\theta}\right] \\
&\quad + \frac{e\beta\kappa_i}{4b^2 - e^2}\mathbb{E}\left[2b^2\left(\mu_\theta + \rho\right) + be\left(\mu_\theta + \rho + \kappa_i p_{i\theta}\left(\theta_i - \mu_\theta\right) + \kappa_i p_{i\rho}\left(1 - \overline{\tau}_\rho\right)\left(\rho - \mu_\rho\right)\right) \middle|\, s_{i\rho}, s_{i\theta}\right].
\end{aligned}$$

Recall that $\mathbb{E}[p_{j1}^{\star} | s_{i\rho}, s_{i\theta}] = p_{j0} + p_{j\theta}\mu_\theta + p_{j\rho}\overline{\tau}_\rho \mathbb{E}[\rho | s_{i\rho}] + p_{j\rho}(1 - \overline{\tau}_\rho)\mu_\rho$. Matching coefficients gives

$$4bp_{i\theta} = 2b - b\beta\kappa_i + \frac{be^2\beta\kappa_i^2 p_{i\theta}}{4b^2 - e^2}; \tag{4}$$

$$4bp_{i\rho} = 2b + 2e\overline{\tau}_\rho p_{j\rho} - b\beta\kappa_i + \frac{e\beta\kappa_i}{4b^2 - e^2}\left(2b^2 + be\left(1 + \left(1 - \overline{\tau}_\rho\right)\kappa_i p_{i\rho}\right)\right). \tag{5}$$

In a symmetric equilibrium, $p_{i\theta} \equiv p_\theta^{\star}$, $p_{i\rho} \equiv p_\rho^{\star}$, and $\kappa_i \equiv \kappa^{\star}$ for both firms $i \in \{1, 2\}$. Then the coefficients in equations (4) and (5) can be solved,

$$\begin{aligned}
p_\theta^{\star} &= \frac{2 - \beta\kappa^{\star}}{4 - \beta^2\kappa^{\star 2}} = \frac{1}{2 + \beta\kappa^{\star}}; \\
p_\rho^{\star} &= \frac{1}{2}\left(1 - \frac{1}{2}\beta\kappa^{\star} + \frac{1}{r}\beta^2\kappa^{\star} + \frac{1}{2}\beta^2\kappa^{\star}\right)\left(1 - \frac{1}{2}r\overline{\tau}_\rho - \frac{1}{4}\left(1 - \overline{\tau}_\rho\right)\beta^2\kappa^{\star 2}\right)^{-1} \\
&= \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa^{\star}}{2 - r\overline{\tau}_\rho - \frac{1}{2}\left(1 - \overline{\tau}_\rho\right)\beta^2\kappa^{\star 2}}.
\end{aligned}$$

The conditional definition of $\kappa^{\star}$ follows from Lemma 4. $\square$

Proof of Theorem 1. The expression for linear price coefficients follows from Lemma 10. Substituting price coefficients into $\kappa^\star$ and letting $\hat{\kappa} \equiv \beta\kappa^\star$ gives

$$\underbrace{(2+\hat{\kappa})^2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2(1-\overline{\tau}_\rho)\,\overline{\tau}_\rho\sigma_\rho^2\hat{\kappa}}_{\text{LHS}(\hat{\kappa})}$$
$$=\underbrace{\left(2-r\overline{\tau}_\rho-\frac{1}{2}(1-\overline{\tau}_\rho)\,\hat{\kappa}^2\right)^2((2+\hat{\kappa})\,\beta-\hat{\kappa})\,\overline{\tau}_\theta\sigma_\theta^2}_{\text{RHS}(\hat{\kappa})}.$$

Note that $\text{LHS}(0) = 0 < \text{RHS}(0)$. Furthermore, we show in Appendix D that $\hat{\kappa} \le r^2/(2-r^2) \equiv \bar{\kappa}$; then we have $\text{LHS}(\bar{\kappa}) > 0 = \text{RHS}(\bar{\kappa})$. Since both LHS and RHS are continuous in $\hat{\kappa}$, it follows that there is a $\hat{\kappa} \in [0,\bar{\kappa}]$ that solves $\text{LHS}(\hat{\kappa}) = \text{RHS}(\hat{\kappa})$.

It is clear that RHS is decreasing in $\hat{\kappa}$, since $(2+\hat{\kappa})\beta + \hat{\kappa} = 2\beta - (1-\beta)\hat{\kappa}$. We now show that LHS is either increasing, or increasing-then-decreasing and concave; in the latter case, we show also that RHS is convex where LHS is decreasing. Since $\text{LHS}(0) < \text{RHS}(0)$ and $\text{LHS}(\bar{\kappa}) > \text{RHS}(\bar{\kappa})$, this is sufficient to show that there is a unique $\hat{\kappa}$ such that $\text{LHS}(\hat{\kappa}) = \text{RHS}(\hat{\kappa})$.

To begin, the derivative of LHS is given by

$$\begin{aligned}
\frac{d\,\text{LHS}}{d\hat{\kappa}} &= 2\,(2+\hat{\kappa})\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2(1-\overline{\tau}_\rho)\,\overline{\tau}_\rho\sigma_\rho^2\hat{\kappa} \\
&\quad -2\left(\frac{1-r}{2-r}\right)(2+\hat{\kappa})^2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)(1-\overline{\tau})\,\overline{\tau}\sigma_\rho^2\hat{\kappa} \\
&\quad +(2+\hat{\kappa})^2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2(1-\overline{\tau})\,\overline{\tau}\sigma_\rho^2 \\
&\propto (2+\hat{\kappa})\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right) \\
&\quad \times\left[2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\hat{\kappa}-2\left(\frac{1-r}{2-r}\right)(2+\hat{\kappa})\,\hat{\kappa}+(2+\hat{\kappa})\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\right].
\end{aligned}$$

The leading terms are positive for $\hat{\kappa} \in [0,\bar{\kappa}]$. The trailing term is

$$\begin{aligned}
&2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\hat{\kappa}-2\left(\frac{1-r}{2-r}\right)(2+\hat{\kappa})\,\hat{\kappa}+(2+\hat{\kappa})\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right) \\
&\qquad \propto -5\,(1-r)\,\hat{\kappa}^2+3r\hat{\kappa}+2\,(2-r)\,. \qquad (6)
\end{aligned}$$

This is a negative quadratic in $\hat{\kappa}$, and is strictly positive when $\hat{\kappa} = 0$; thus LHS is either increasing for $\hat{\kappa} \in [0,\bar{\kappa}]$, or it is increasing-then-decreasing on this range. When goods are

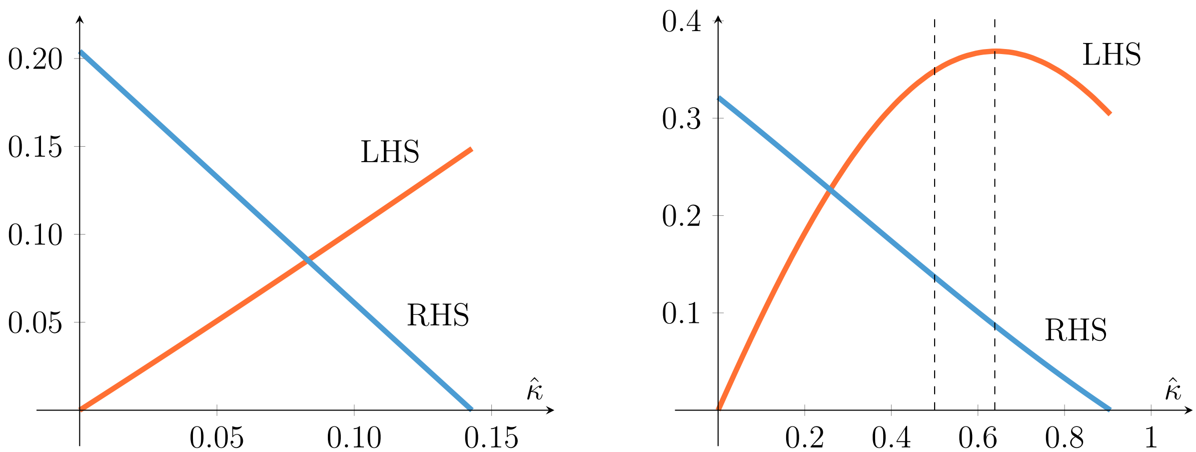

Figure 5: A graphical depiction of the proof of equilibrium existence and uniqueness. The existence of an equilibrium amounts to finding a $\hat{\kappa}$ such that $\text{LHS}(\hat{\kappa}) = \text{RHS}(\hat{\kappa})$. Since $\text{LHS}(0) < \text{RHS}(0)$ and $\text{LHS}(\overline{\kappa}) > \text{RHS}(\overline{\kappa})$ and both functions are continuous, such a $\hat{\kappa}$ is guaranteed to exist. For all parameter specifications RHS is decreasing. We show that either LHS is increasing (left panel) or increasing and then decreasing (right panel). In the former case, it is clear that there is a unique point of intersection and hence a unique equilibrium. In the latter case, we show that LHS is concave where it is decreasing and RHS is convex anywhere LHS is decreasing. Then $\text{LHS} - \text{RHS}$ is concave, ensuring that equilibrium $\hat{\kappa}$ is unique. Plot ranges differ, since the upper bound on $\hat{\kappa}$, $\overline{\kappa} = r^2/(2 - r^2)$ depends on the parameter r. Dashed lines in the right panel appear at $\hat{\kappa} = 1/2$ and $\hat{\kappa} = (\sqrt{249} - 3)/20$, the bounds applied in the proof.

substitutes ($r \geq 0$) this is positive for all relevant $\hat{\kappa}$ and the proof is complete. We then focus on the case where goods are complements ($r < 0$).

Replacing the leading positive terms in LHS gives

$$\frac{d\,\mathrm{LHS}}{d\hat{\kappa}} \propto \left(2\left(2-r\right)+3r\hat{\kappa}-5\left(1-r\right)\hat{\kappa}^2\right)\left(2\left(2-r\right)+r\hat{\kappa}-\left(1-r\right)\hat{\kappa}^2\right).$$

This implies

$$\frac{d^2\,\mathrm{LHS}}{d\hat{\kappa}^2} \propto 8\left(2-r\right)r-6\left(3r^2-12r+8\right)\hat{\kappa}-24\left(1-r\right)r\hat{\kappa}^2+20\left(1-r\right)^2\hat{\kappa}^3.$$

This is negative at $\hat{\kappa} = 0$ and $\hat{\kappa} = 1 \geq \bar{\kappa}$. Moreover,

$$\frac{d^3\,\mathrm{LHS}}{d\hat{\kappa}^3} \propto -\left(18r^2-72r+48\right)-48\left(1-r\right)r\hat{\kappa}+60\left(1-r\right)^2\hat{\kappa}^2.$$

This is a positive quadratic in $\hat{\kappa}$, thus $d^2\,\mathrm{LHS}/d\hat{\kappa}^2$ is either decreasing, decreasing-then-increasing, or increasing for $\hat{\kappa} \in [0, \bar{\kappa}]$. Since $d^2\,\mathrm{LHS}/d\hat{\kappa}^2 \leq 0$ for $\hat{\kappa} \in \{0, \bar{\kappa}\}$, it follows that $d^2\,\mathrm{LHS}/d\hat{\kappa}^2 \leq 0$ for all $\hat{\kappa} \in [0, \bar{\kappa}]$, and LHS is concave.

If LHS is decreasing, it must be that the quadratic in (6) is negative. The zeros of this quadratic are given by

$$\hat{\kappa}_{\pm} \in \frac{3r}{10-10r} \pm \frac{1}{10-10r}\sqrt{9r^2+8\left(2-r\right)\left(5-5r\right)} = \frac{1}{10-10r}\left(3r+\sqrt{49r^2-120r+80}\right).$$

LHS is decreasing only if goods are complements, $r < 0$, so only the "+" solution is valid. Note that

$$\begin{aligned}\frac{d\hat{\kappa}_+}{dr} &\overset{\text{sign}}{=} \left(3+\frac{49r-60}{\sqrt{49r^2-120r+80}}\right)\left(10-10r\right)+10\left(3r+\sqrt{49r^2-120r+80}\right)\\ &\overset{\text{sign}}{=} 20-11r+3\sqrt{49r^2-120r+80} > 0.\end{aligned}$$

Then $\hat{\kappa}_+$ is minimized when $r = -1$ (since $r \in [-1, 1]$ and $d\hat{\kappa}_+/dr < 0$ when $r < 0$). This gives that if LHS is decreasing at $\hat{\kappa}$,

$$\hat{\kappa} \geq \bar{\kappa}_+ = \frac{1}{20}\left(-3+\sqrt{249}\right) \geq \frac{15-3}{20} > \frac{1}{2}.$$

Finally, we compute the second derivative of RHS with respect to $\hat{\kappa}$ to show that RHS is convex,

$$\begin{aligned}\frac{d^2\,\mathrm{RHS}}{d\hat{\kappa}^2} &= \frac{d}{d\hat{\kappa}}\Bigg[-2\left(1-\overline{\tau}_\rho\right)\hat{\kappa}\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)\left(2\beta-\left(1-\beta\right)\hat{\kappa}\right)\\ &\qquad\qquad -\left(1-\beta\right)\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)^2\Bigg]\\ &= 2\left(1-\overline{\tau}_\rho\right)\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)\left(4\left(1-\beta\right)\hat{\kappa}-4\beta\right)\\ &\quad +4\left(1-\overline{\tau}_\rho^2\right)\hat{\kappa}^2\left(2\beta-\left(1-\beta\right)\hat{\kappa}\right).\end{aligned}$$

Note that all involved terms are positive for $\hat{\kappa}\in[0,\bar{\kappa}]$, with the potential exception of $4(1-\beta)\hat{\kappa}-\beta$. As shown above, $\hat{\kappa}\geq 1/2$ whenever LHS is decreasing, hence

$$4\left(1-\beta\right)\hat{\kappa}-4\beta\geq 2\left(1-\beta\right)-4\beta=2-6\beta\geq 0. \qquad \text{(since } \beta=r^2/(4-r^2)\leq 1/3\text{)}$$

Then $d^2\,\mathrm{RHS}/d\hat{\kappa}^2>0$ when LHS is decreasing. Then where LHS is decreasing it is convex and RHS is concave, implying a unique intersection. □

Proof of Proposition 1. When $e=0$, $\beta=e^2/(4b^2-e^2)=0$. Then $p^\star_\theta=p^\star_\rho=1/2$. Otherwise, we compare

$$\begin{aligned} p^\star_\theta \gtrless p^\star_\rho \iff &\qquad \frac{1}{2+\beta\kappa^\star}\gtrless\frac{1-\left(\frac{b-e}{2b-e}\right)\beta\kappa^\star}{2-\frac{e}{b}\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^{\star 2}}\\ \iff &\qquad 2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^{\star 2}\gtrless\left(2+\beta\kappa^\star\right)\left(1-\left(\frac{1-r}{2-r}\right)\beta\kappa^\star\right)\\ \iff &\qquad -r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^{\star 2}\gtrless\left(\frac{r}{2-r}\right)\beta\kappa^\star-\left(\frac{1-r}{2-r}\right)\beta^2\kappa^{\star 2}. \end{aligned} \tag{7}$$

When $r>0$ the left-hand side of (7) is maximized when $\overline{\tau}_\rho=0$. This leads to

$$p^\star_\theta<p^\star_\rho \quad\Longleftarrow\quad \left(\frac{1-r}{2-r}-\frac{1}{2}\right)\beta\kappa^\star<\frac{r}{2-r}.$$

The left-hand side is negative and the right-hand side is positive, so $p^\star_\theta<p^\star_\rho$.

When $r<0$ the left-hand side of (7) is minimized when $\overline{\tau}_\rho=0$. This leads to

$$p^\star_\theta>p^\star_\rho \quad\Longleftarrow\quad \left(\frac{1-r}{2-r}-\frac{1}{2}\right)\beta\kappa^\star>\frac{r}{2-r}.$$

The left-hand side is positive and the right-hand side is negative, so $p^\star_\theta>p^\star_\rho$. □

Proof of Proposition 2. The inverse relationship of $p_\theta^\star$ and $\kappa^\star$ follows immediately from the definition $p_\theta^\star = 1/(2+\beta\kappa^\star)$.

The remaining relationships follow from the quintic implicit equation for $\kappa^\star$,

$$\underbrace{(2+\beta\kappa^\star)^2\left(1-\left(\frac{1-r}{2-r}\right)\beta\kappa^\star\right)^2\sigma_{i\rho}^2\overline{\tau}_\rho^2\kappa^\star}_{\text{LHS}(\kappa^\star)}$$
$$=\underbrace{\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^{\star 2}\right)^2\left(2-\left(1-\beta\right)\kappa^\star\right)\sigma_\theta^2\overline{\tau}_\theta}_{\text{RHS}(\kappa^\star)}.\text{[37]}$$

Note that LHS is constant in $\overline{\tau}_\theta$ and RHS is linearly increasing in $\overline{\tau}_\theta$. All involved functions are continuous and differentiable, hence we check

$$\frac{d}{d\overline{\tau}_\theta}\left[\text{LHS}\left(\kappa^\star\right)-\text{RHS}\left(\kappa^\star\right)\right]=\left(\frac{\partial\text{LHS}}{\partial\kappa^\star}-\frac{\partial\text{RHS}}{\partial\kappa^\star}\right)\frac{\partial\kappa^\star}{\partial\overline{\tau}_\theta}+\left(\frac{\partial\text{LHS}}{\partial\overline{\tau}_\theta}-\frac{\partial\text{RHS}}{\partial\overline{\tau}_\theta}\right).$$

Then

$$\frac{\partial\kappa^\star}{\partial\overline{\tau}_\theta}=\frac{\frac{\partial\text{RHS}}{\partial\overline{\tau}_\theta}}{\frac{\partial\text{LHS}}{\partial\kappa^\star}-\frac{\partial\text{RHS}}{\partial\kappa^\star}}. \tag{8}$$

At the unique $\kappa^\star$ such that LHS($\kappa^\star$) = RHS($\kappa^\star$) it is the case that $\partial\text{LHS}(\kappa^\star)/\partial\kappa^\star > \partial\text{RHS}(\kappa^\star)/\partial\kappa^\star$, it follows that $\kappa^\star$ is increasing in $\overline{\tau}_\theta$.

To compute comparative statics with respect to $\overline{\tau}_\rho$, we first check

$$\begin{aligned}\frac{2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^{\star 2}}{\overline{\tau}_\rho\sqrt{\sigma_{i\rho}^2}}&=\frac{1}{\sqrt{\sigma_{i\rho}^2}}\left(\frac{1}{\overline{\tau}_\rho}\left(2-\frac{1}{2}\beta^2\kappa^{\star 2}\right)-\left(r-\frac{1}{2}\beta^2\kappa^{\star 2}\right)\right)\\&=\left(\left(\frac{\sqrt{\sigma_{i\rho}^2}}{\sigma_\rho^2}+\frac{1}{\sqrt{\sigma_{i\rho}^2}}\right)\left(2-\frac{1}{2}\beta^2\kappa^{\star 2}\right)-\frac{1}{\sqrt{\sigma_{i\rho}^2}}\left(r-\frac{1}{2}\beta^2\kappa^{\star 2}\right)\right)\\&=\frac{1}{\sqrt{\sigma_\rho^2}}\left(\sqrt{\frac{\sigma_{i\rho}^2}{\sigma_\rho^2}}\left(2-\frac{1}{2}\beta^2\kappa^{\star 2}\right)+\sqrt{\frac{\sigma_\rho^2}{\sigma_{i\rho}^2}}\left(2-r\right)\right).\end{aligned}$$

Fixing σ_ρ^2, $\overline{\tau}_\rho$ increases when $\sigma_{i\rho}^2$ decreases (and vice versa). Letting $R_\rho=\sqrt{\sigma_{i\rho}^2/\sigma_\rho^2}$, we define

[37] The functions LHS and RHS differ slightly from those used in the proof of Theorem 1. In particular, they are functions of $\kappa^\star$ and include β terms, while those used in the proof of Theorem 1 are functions of $\hat{\kappa}$ and do not include β terms.

LHS^R and RHS^R as

$$\underbrace{(2+\beta\kappa^\star)^2\left(1-\left(\frac{1-r}{2-r}\right)\beta\kappa^\star\right)^2\kappa^\star}_{\text{LHS}^R(\kappa^\star)}$$

$$=\underbrace{\frac{1}{\sqrt{\sigma_\rho^2}}\left(\left(2-\frac{1}{2}\beta^2\kappa^{\star 2}\right)R_\rho+(2-r)\frac{1}{R_\rho}\right)(2-(1-\beta)\kappa^\star)\,\sigma_\theta^2\overline{\tau}_\theta\,.}_{\text{RHS}^R(\kappa^\star)}$$

Note that LHS^R is constant in R_ρ. Holding $\kappa^\star$ fixed, the extent to which RHS^R is affected by R_ρ is given by

$$\frac{\partial}{\partial R_\rho}\left[\left(2-\frac{1}{2}\beta^2\kappa^{\star 2}\right)R_\rho+(2-r)\frac{1}{R_\rho}\right]=\left(2-\frac{1}{2}\beta^2\kappa^{\star 2}\right)-\frac{2-r}{R_\rho^2}. \tag{9}$$

Since $\beta\kappa^\star\leq 1$ and $r\leq 1$, the above is negative when R_ρ^2 is small and positive when R_ρ^2 is large. Since $R_\rho^2=(1-\overline{\tau}_\rho)/\overline{\tau}_\rho$, the above is negative when $\overline{\tau}_\rho$ is large and positive when $\overline{\tau}_\rho$ is small. An analysis similar to equation (8) implies that $\kappa^\star$ is decreasing in R_ρ (increasing in $\overline{\tau}_\rho$) when $\overline{\tau}_\rho$ is large and increasing in R_ρ (decreasing in $\overline{\tau}_\rho$) when $\overline{\tau}_\rho$ is small. To see single-peakedness, note that as R_ρ increases, (9) also increases. Starting from a point at which $\kappa^\star$ is locally constant, a slight increase in R_ρ from a point at which $\kappa^\star$ is locally constant must cause $\kappa^\star$ to rise; otherwise, $\kappa^\star$ is falling, implying that (9) is even more positive, a contradiction.

Finally, note that

$$\left(2-\frac{1}{2}\beta^2\kappa^{\star 2}\right)-\frac{2-r}{R_\rho^2}=2\left(\frac{R_\rho^2-1}{R_\rho^2}\right)+\left(\frac{r-\frac{1}{2}\beta^2\kappa^{\star 2}R_\rho^2}{R_\rho^2}\right).$$

When $R_\rho^2=1$, this is simply $r-\beta^2\kappa^{\star 2}/2\overset{\text{sign}}{=}r$. Then (9) is positive at $R_\rho^2=1$ ($\overline{\tau}_\rho=1/2$) when $r>0$, and negative at $R_\rho^2=1$ when $r<0$. From single-peakedness, it follows that $\kappa^\star$ is minimized at $\overline{\tau}^\star>1/2$ when $r>0$ and at $\overline{\tau}^\star<1/2$ when $r<0$. □

Lemma 11. *In the n-firm extension expected second period prices are given by*

$$\mathbb{E}\left[p_{i2n}^\star\middle|\,\rho,\mathbf{p}_1\right]=\frac{a}{2b-e}+(n-1)\left(\frac{1}{2\,(n-1)\,b+e}\right)\mathbb{E}\left[bc_i\middle|\,\rho,\mathbf{p}_1\right]$$
$$+\frac{e}{(2\,(n-1)\,b+e)\,(2b-e)}\sum_{j=1}^{n}\mathbb{E}\left[bc_j\middle|\,\rho,\mathbf{p}_1\right].$$

Proof. Firm i's second period objective is

$$\max_p \mathbb{E}\left[\frac{1}{n-1}\left(a - bp + \frac{e}{n-1}\sum_{j\neq i} p^\star_{j2n}\right)(p-c_i)\middle| c_i, \mathbf{p}_1\right].$$

At the optimal price $p^\star_{i2n}$, firm i's second period first-order condition is

$$0 = a - 2bp^\star_{i2n} + bc_i + \frac{e}{n-1}\sum_{j\neq i}\mathbb{E}\left[p^\star_{j2n}\middle|\rho, \mathbf{p}_1\right].$$

This equation holds given firm i's second period information; therefore it holds in expectation, conditional on ρ and $\mathbf{p}_1$. This gives

$$0 = a - 2b\mathbb{E}\left[p^\star_{i2n}\middle|\rho, \mathbf{p}_1\right] + b\mathbb{E}\left[c_i\middle|\rho, \mathbf{p}_1\right] + \frac{e}{n-1}\sum_{j\neq i}\mathbb{E}\left[p^\star_{j2n}\middle|\rho, \mathbf{p}_1\right].$$

Taken over all firms i this is a linear system, $A\mathbb{E}[\mathbf{p}^\star_2|\rho, \mathbf{p}_1] = a + b\mathbb{E}[\mathbf{c}|\rho, \mathbf{p}_1]$, where

$$A_{ii} = 2b, \quad A_{ij} = -\frac{e}{n-1} \quad (j \neq i).$$

The matrix A is invertible, with

$$A^{-1}_{ii} = \frac{2(n-1)b - (n-2)e}{(2b-e)(2(n-1)b+e)}, \quad A^{-1}_{ij} = \frac{e}{(2b-e)(2(n-1)b+e)} \quad (j \neq i).$$

This implies the stated result. □

Corollary 4. *In the n-firm extension second period prices are given by*

$$p^\star_{i2n} = \frac{a}{2b-e} + \frac{1}{2}\left(c_i + \frac{e^2}{(2(n-1)b+e)(2b-e)}\mathbb{E}\left[c_i\middle|\rho, \mathbf{p}_1\right]\right) + \sum_{j\neq i}\frac{be}{(2(n-1)b+e)(2b-e)}\mathbb{E}\left[c_j\middle|\rho, \mathbf{p}_1\right].$$

Proof. This follows immediately from the proof of Lemma 11. □

Lemma 12. *In the n-firm extension expected second period profits are given by*

$$\mathbb{E}\left[\pi^\star_{i2n}\middle| c_i, \mathbf{p}_1\right] = \frac{1}{4b(n-1)}\left(a - bc_i + \frac{e}{n-1}\sum_{j\neq i}\mathbb{E}\left[p^\star_{j2n}\middle|\rho, \mathbf{p}_1\right]\right)^2$$

Proof. This is a standard profit-maximization problem, and follows immediately from the optimization in Lemma 11. □

Proof of Theorem 2. Firm i's first period objective is

$$\max_p \mathbb{E}\left[\frac{1}{n-1}\left(a - bp + \frac{e}{n-1}\sum_{j\neq i} p^\star_{j1n}\right)(p - c_i) + \pi^\star_{i2n}(p)\middle| s_i\right].$$

Following Lemma 12, the firm's first-order condition is

$$\begin{aligned}2bp^\star_{i1n} = {} & \mathbb{E}\left[a + bc_i + \frac{e}{n-1}\sum_{j\neq i} p^\star_{j1n}\middle| s_i\right] \\ & + \frac{e}{2b}\mathbb{E}\left[\left(a - bc_i + \frac{e}{n-1}\sum_{j\neq i} p^\star_{j2n}\right)\left(\frac{1}{n-1}\sum_{j\neq i}\frac{dp^\star_{j2n}}{dp_{i1n}}\right)\middle| s_i\right].\end{aligned} \tag{10}$$

Conditional on ρ, firm i's price does not affect firm j's beliefs about firm k's price. Then following Corollary 4,

$$\frac{d}{dp_{i1n}}\mathbb{E}\left[p^\star_{j2n}\middle| \rho, \mathbf{p}_1\right] = \frac{be}{(2(n-1)b + e)(2b - e)}\left(\frac{d}{dp_{i1n}}\mathbb{E}\left[c_i\middle| \rho, \mathbf{p}_1\right]\right).$$

Conditional on ρ, the effect of firm i's first period price on firm j's beliefs about i's costs is completely determined by the relative importance of private and public costs in setting first period prices. That is,

$$\frac{d}{dp_{i1n}}\mathbb{E}\left[c_i\middle| \rho, \mathbf{p}_1\right] = \kappa^\star_n \equiv \frac{\sigma^2_\theta \overline{\tau}_\theta p_{\theta n}}{\sigma^2_\rho(1 - \overline{\tau}_\rho)\overline{\tau}_\rho p^2_{\rho n} + \sigma^2_\theta \overline{\tau}_\theta p^2_{\theta n}}.$$

Define $\beta_n = e^2/(2(n-1)b+e)(2b-e)$; note that $\beta_2 = \beta$ as defined in the base two-firm case. Then equation (10) becomes, for any $j \neq i$,

$$2bp^\star_{i1n} = \left(1 + \frac{e}{2b}\right)a + \left(1 - \frac{1}{2}\beta_n\kappa^\star_n\right)\mathbb{E}\left[bc_i\middle| s_i\right] + e\mathbb{E}\left[p^\star_{j1n}\middle| s_i\right] + \frac{1}{2}e\beta_n\kappa^\star_n\mathbb{E}\left[p^\star_{j2n}\middle| s_i\right].$$

Corollary 4 implies[38]

$$\begin{aligned}\mathbb{E}\left[p^\star_{j2n}\middle| s_i\right] &= \frac{a}{2b-e} + \frac{1}{2}(1 + \beta_n)\mathbb{E}\left[c_j\middle| s_i\right] + \sum_{k\neq i,j}\frac{b}{e}\beta_n\mathbb{E}\left[c_k\middle| s_i\right] + \frac{b}{e}\beta_n\mathbb{E}\left[\mathbb{E}\left[c_i\middle| \rho, \mathbf{p}_1\right]\middle| s_i\right] \\ &= \frac{a}{2b-e} + \left(\frac{1}{2}(1 + \beta_n) + (n-2)\frac{b}{e}\beta_n\right)\mathbb{E}\left[c_j\middle| s_i\right] + \frac{b}{e}\beta_n\mathbb{E}\left[\mathbb{E}\left[c_i\middle| \rho, \mathbf{p}_1\right]\middle| s_i\right].\end{aligned}$$

[38]Note that $b\beta_n/e = be/(2(n-1)b+e)(2b-e)$, so divding by zero is not a relevant concern.

In the linear pricing equilibrium,

$$p_{i1n}^{\star} = p_{0n} + p_{\theta n}\mathbb{E}\left[\theta_i | \, s_i\right] + p_{\rho n}\mathbb{E}\left[\rho | \, s_i\right].$$

Matching coefficients gives

$$\begin{aligned}
2bp_{\theta n} &= b + \frac{e}{2b}\left(-b + b\beta_n\kappa_n^{\star}p_{\theta n}\right)\frac{b}{e}\beta_n\kappa \\
&= \left(b - \frac{1}{2}b\beta_n\kappa_n^{\star}\right) + \frac{1}{2}b\beta_n^2\kappa_n^{\star 2}p_{\theta n}; \\
2bp_{\rho n} &= b + e\overline{\tau}_\rho p_{\rho n} \\
&\quad + \left(\frac{e}{2b}\right)\left(-b + \left(\frac{1}{2}\left(1+\beta_n\right) + (n-2)\frac{b}{e}\beta_n + \frac{b}{e}\beta_n + \frac{b}{e}\beta_n\kappa_n^{\star}p_{\rho n}\left(1-\overline{\tau}_\rho\right)\right)e\right)\frac{b}{e}\beta_n\kappa \\
&= b + e\overline{\tau}_\rho p_{\rho n} + \frac{1}{2}\left(-b + \frac{1}{2}\left(1+\beta_n\right)e + (n-1)\,b\beta_n + b\beta_n\kappa_n^{\star}p_{\rho n}\left(1-\overline{\tau}_\rho\right)\right)\beta_n\kappa.
\end{aligned}$$

The stated equalities are immediate. □

Proof of Corollary 1. The expressions for $p_{\theta\infty}$ and $p_{\rho\infty}$ follow immediately from Theorem 2.

As mentioned in the main text, $\lim_{n \nearrow \infty} \beta_n = 0$. In the n-large limit information about firm i does not affect any firm j's second period pricing strategy. Then firm i's first period first order conditions (equation (10) in the proof of Theorem 2 above) reduce to[39]

$$2bp_{i1\infty}^{\star} = a + b\mathbb{E}\left[c_i | \, s_i\right] + e\mathbb{E}\left[p_{j1\infty}^{\star} \middle| \, s_i\right] \quad \text{for any } j \neq i.$$

In expectation this is

$$2b\mathbb{E}\left[p_{i1\infty}^{\star}\right] = a + b\mathbb{E}\left[\mathbb{E}\left[c_i | \, s_i\right]\right] + e\mathbb{E}\left[\mathbb{E}\left[p_{j1\infty}^{\star} \middle| \, s_i\right]\right] = a + b\mathbb{E}\left[c_i\right] + e\mathbb{E}\left[p_{j1\infty}^{\star}\right].$$

In the linear equilibrium this implies

$$2b\left(p_{0\infty} + p_{\theta\infty}\mu_\theta + p_{\rho\infty}\mu_\rho\right) = a + b\left(\mu_\theta + \mu_\rho\right) + e\left(p_{0\infty} + p_{\theta\infty}\mu_\theta + p_{\rho\infty}\mu_\rho\right).$$

Algebraic rearrangement gives

$$\left(2b - e\right)p_{0\infty} = a + b\left(\mu_\theta + \mu_\rho\right) - \left(2b - e\right)p_{\theta\infty}\mu_\theta - \left(2b - e\right)p_{\rho\infty}\mu_\rho.$$

Substituting in for $p_{\theta\infty}$ and $p_{\rho\infty}$ yields the stated equation for $p_{0\infty}$. □

[39] Recall that $\mathbb{E}[\sum_{k\neq i} p_{k1n}^{\star}|s_i]/(n-1) = \mathbb{E}[p_{j1}^{\star}|s_i]$ for any $j \neq i$.

B Proofs for Section 4

To simplify notation in this appendix, we use $\simeq_x$ to denote an equivalence of all terms that depend directly on the parameter x; that is, $f(\cdot) \simeq g(\cdot)$ if there is $C \in \mathbb{R}$ such that for any $x \in \operatorname{Supp} x$, $g(x) - f(x) = C$.

Proof of Lemma 5. Second period prices p_{i2}^c follow from the same methodology applied in the proof of Lemma 1. First period prices p_{i1}^c follow from the same methodology applied in the proof of Lemma 3. □

Lemma 13. *When common cost information is shared, expected second period prices are*

$$\mathbb{E}\left[p_{i2}^c \middle| \rho, s_\rho, \mathbf{p}_1\right] = \frac{1}{4b^2 - e^2}\left((2b+e)\,a + 2b^2\mathbb{E}\left[c_i \middle| \rho, s_\rho, \mathbf{p}_1\right] + be\mathbb{E}\left[c_j \middle| \rho, s_\rho, \mathbf{p}_1\right]\right).$$

Proof. Following Lemma 5, we have

$$\mathbb{E}\left[p_{i2}^c \middle| \rho, s_\rho, \mathbf{p}_1\right] = \frac{1}{2b}\left(a + b\mathbb{E}\left[c_i \middle| \rho, s_\rho, \mathbf{p}_1\right] + e\mathbb{E}\left[p_{j2}^c \middle| \rho, s_\rho, \mathbf{p}_1\right]\right).$$

This yields two linear equations in two unknowns. Solving this linear system gives the desired equation. □

Proof of Proposition 3. We begin by computing p_θ^c, then address comparisons to the no-information-sharing regime.

Lemma 5 and the statement that $\partial\mathbb{E}[p_{j2}^c|\rho, s_\rho, \mathbf{p}_1]/dp_{i1} = b\beta/ep_{i\theta}^c$ give that first period prices are

$$p_{i1}^c = \frac{1}{2b}\mathbb{E}\left[a + bc_i + ep_{j1}^c \middle| s_i\right] + \frac{1}{4b}\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}^c \middle| \rho, s_\rho, \mathbf{p}_1\right]\right)\frac{\beta}{p_{i\theta}^c}\middle| s_i\right]. \tag{11}$$

Following Lemma 13 we have

$$\begin{aligned}
&\mathbb{E}\left[-bc_i + e\mathbb{E}\left[p_{j2}^c \middle| \rho, s_\rho, \mathbf{p}_1\right] \middle| s_i\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[-bc_i + ep_{j2}^c \middle| \rho, s_\rho, \mathbf{p}_1\right] \middle| s_i\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[-bc_i + \frac{e}{4b^2 - e^2}\left((2b+e)\,a + 2b^2c_j + bec_i\right)\middle| \rho, s_\rho, \mathbf{p}_1\right]\middle| s_i\right] \\
&= \frac{1}{4b^2 - e^2}\mathbb{E}\left[\mathbb{E}\left[(2b+e)\,ea + 2b^2ec_j - 2\left(2b^2 - e^2\right)bc_i \middle| \rho, s_\rho, \mathbf{p}_1\right] \middle| s_i\right].
\end{aligned}$$

In a linear equilibrium, $s_{i\theta}$ is perfectly revealed by p_{i1}. Then the above is

$$\begin{aligned}
&\mathbb{E}\left[-bc_i + e\mathbb{E}\left[p^c_{j2}\middle|\,\rho, s_\rho, \mathbf{p}_1\right]\middle|\, s_i\right] \\
&= \frac{1}{4b^2-e^2}\mathbb{E}\left[(2b+e)\,ea - 2\left(2b^2-e^2\right)bc_i + 2b^2e\mathbb{E}\left[c_j\middle|\,\rho, s_\rho, \mathbf{p}_1\right]\middle|\, s_i\right].
\end{aligned}$$

In the linear equilibrium, $p_{i1} = p^c_{i0} + p^c_{i\theta}\mathbb{E}\left[\theta_i|\, s_i\right] + p^c_{i\rho}\mathbb{E}\left[\rho|\, s_i\right]$. Restricting equation (11) to terms which depend on $\mathbb{E}[\theta_i|s_i]$ gives

$$p^c_{i\theta} = \frac{1}{2} - \frac{1}{2}\left(\frac{2b^2-e^2}{4b^2-e^2}\right)\frac{\beta}{p_{i\theta c}} = \frac{1}{2} + \frac{1}{4}\left(\frac{(\beta-1)\,\beta}{p^c_{i\theta}}\right) \quad\Longrightarrow\quad 4\left[p^c_{i\theta}\right]^2 - 2p^c_{i\theta} - (\beta-1)\,\beta = 0.$$

The solutions of this quadratic are

$$\begin{aligned}
p^c_{i\theta} &= \frac{1}{8}\left(2 \pm \sqrt{4+16\,(\beta-1)\,\beta}\right) \\
&= \frac{1}{4} \pm \frac{1}{4}\sqrt{4\beta^2-4\beta+1} = \frac{1}{4}\left(1 \pm (2\beta-1)\right) \in \left\{-\frac{1}{2}\beta, \frac{1}{2}\,(1-\beta)\right\}.
\end{aligned}$$

Since $\beta = e^2/(4b^2-e^2) \geq 0$, one solution is positive and the other is negative.[40] Then $p^c_{i\theta} = p^c_\theta = (1-\beta)/2$ for both firms.

Recall that $p^\star_\theta = 1/(2+\beta\kappa^\star)$. By its definition in Lemma 4, $\kappa^\star \leq 1/p_\theta = 2+\beta\kappa^\star$; then $\kappa^\star \leq 2/(1-\beta)$. It follows that

$$p^\star_\theta \geq \frac{1}{2+\frac{2\beta}{1-\beta}} = \frac{1-\beta}{2} = p^c_\theta.$$

The inequality $\kappa^\star \leq 1/p_\theta$ is strict whenever $\beta > 0$ and $\sigma^2_\rho(1-\overline{\tau}_\rho)\tau_\rho p^2_\rho > 0$; since we have assumed signals are informative, this is true whenever $e \neq 0$.

The inequality $p^c_\theta \leq p^\star_\theta$ implies

$$\beta\kappa^\star = \frac{1-2p^\star_\theta}{p^\star_\theta} \leq \frac{1-2p^c_\theta}{p^c_\theta} = \frac{\beta}{p^c_\theta} = \beta\kappa^c \quad\Longleftrightarrow\quad \kappa^c \geq \kappa^\star.$$

$\square$

[40] When $e = 0$, $-\beta/2 = 0$. This solution can be ruled out by second order conditions, but we omit this exercise: if $p^c_{i\theta} = 0$, prices do not depend on private cost information, which is not possible in our equilibrium.

Proof of Theorem 3. Following from Lemma 2 ex-ante expected second period prices are

$$\begin{aligned}
\mathbb{E}[p^\star_{j2}] &= \frac{1}{4b^2-e^2}\left((2b+e)\,a + 2b^2\mathbb{E}\left[\mathbb{E}\left[c_j|\,\rho, p_{j1}\right]\right] + be\mathbb{E}\left[\mathbb{E}\left[c_i|\,\rho, p_{i1}\right]\right]\right) \\
&= \frac{1}{4b^2-e^2}\left(\mathbb{E}\left[(2b+e)a + (2b^2+be)\left(\mu_\theta + \mathbb{E}\left[\rho|s_{i,\rho}\right]\right)\right]\right. \\
&\quad \left. + be\mathbb{E}\left[\kappa^\star\left(\mathbb{E}\left[p^\star_{i1}|\,s\right] - \left(p^\star_0 + p^\star_\theta\mu_\theta + p^\star_\rho\mathbb{E}\left[\rho|\,s\right]\right)\right)\right]\right)
\end{aligned}$$

The latter term equals zero, $\mathbb{E}[\mathbb{E}[p^\star_{i1}|s_i] - (p^\star_0 + p^\star_\theta\mu_\theta + p^\star_\rho\mathbb{E}[\rho|s_i])] = 0$. It follows that in equilibrium

$$\mathbb{E}[p^\star_{j2}] = \frac{a + b(\mu_\rho + \mu_\theta)}{2b-e}.$$

Similarly, with information sharing

$$\begin{aligned}
\mathbb{E}[p^c_{j2}] &= \frac{1}{4b^2-e^2}\left((2b+e)\,a + 2b^2\mathbb{E}\left[\mathbb{E}\left[c_j|\,\rho, s_\rho, p_{j1}\right]\right] + be\mathbb{E}\left[\mathbb{E}\left[c_i|\,\rho, s_\rho, p_{i1}\right]\right]\right) \\
&= \frac{1}{4b^2-e^2}\left((2b+e)\,a + 2b^2\mathbb{E}\left[\rho + \mathbb{E}[\theta_j|s_{j\theta}]\right] + be\mathbb{E}\left[\rho + \mathbb{E}[\theta_i|s_{i\theta}]\right]\right) \\
&= \frac{a + b(\mu_\rho + \mu_\theta)}{2b-e}.
\end{aligned}$$

In equilibrium the ex-ante expected price for each firm in the second period are the same with and without information sharing.

From Lemma 3 the first period price in the symmetric equilibrium is

$$\begin{aligned}
\mathbb{E}[p^\star_{i1}] &= \frac{1}{2b-e}\mathbb{E}\left[\mathbb{E}\left[bc_i + a + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[p^\star_{j2}\middle|\,\rho, s_{i\rho}, p_{j1}\right]\right)\frac{be}{4b^2-e^2}\kappa^\star\middle|\, s_{i\rho}, s_{i\theta}\right]\right] \\
&= \frac{1}{2b-e}\left(a + b(\mu_\rho + \mu_\theta) + \frac{e^2\kappa^\star}{2(4b^2-e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p^\star_{j2}\middle|\,\rho, s_{i,\rho}, p_{j1}\right]\right)\middle|\, s_{i\rho}, s_{i\theta}\right]\right]\right).
\end{aligned}$$

Similarly, from Lemma 5 when firms are sharing common cost information expected first period prices are

$$\begin{aligned}
\mathbb{E}[p^c_{i1}] &= \frac{1}{2b-e}\mathbb{E}\left[bc_i + a + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[p^c_{j2}\middle|\,\rho, s_\rho, p_{j1}\right]\right)\frac{be}{4b^2-e^2}\kappa^c\middle|\, s_\rho, s_{i\theta}\right] \\
&= \frac{1}{2b-e}\left(a + b(\mu_\rho + \mu_\theta) + \frac{e^2\kappa^c}{2(4b^2-e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p^c_{j2}\middle|\,\rho, s_\rho, p_{j1}\right]\right)\middle|\, s_\rho, s_{i\theta}\right]\right]\right).
\end{aligned}$$

All terms are identical except $\kappa^\star$ and κ^c. From Proposition 3, $\kappa^c \geq \kappa^\star$, where the inequality is strict when $e \neq 0$. Therefore $\mathbb{E}[p^c_{i1}] \geq \mathbb{E}[p^\star_{i1}]$, where the inequality is strict when $e \neq 0$. $\square$

Lemma 14. *The demand specification* (1) *is generated by the utility function*

$$u(\mathbf{q};\mathbf{p}) = \frac{a}{b-e}\sum_{i=1}^{n} q_i - \frac{n-1}{2}\left(\frac{(n-1)\,b-(n-2)\,e}{((n-1)\,b+e)\,(b-e)}\right)\sum_{i=1}^{n} q_i^2$$
$$-\frac{n-1}{2}\left(\frac{e}{((n-1)\,b+e)\,(b-e)}\right)\sum_{i=1}^{n}\sum_{j\neq i} q_i q_j - \sum_{i=1}^{n} p_i q_i. \qquad (12)$$

In particular, when $n = 2$ the utility specification in (2) *generates demand specification in Section 4.*

Proof. Fix a price vector $\mathbf{p}$. Given utility as in (12), the consumer's first-order condition with respect to quantity q_i is

$$\frac{a}{b-e} - (n-1)\left(\frac{(n-1)\,b-(n-2)\,e}{((n-1)\,b+e)\,(b-e)}\right) q_i - (n-1)\left(\frac{e}{((n-1)\,b+e)\,(b-e)}\right)\sum_{j\neq i} q_j = p_i.$$

Let $Q = \sum_{i=1}^{n} q_i$. Summing up both sides of the equation over all firms i gives

$$\frac{na}{b-e} - (n-1)\left(\frac{(n-1)\,b-(n-2)\,e}{((n-1)\,b+e)\,(b-e)}\right) Q - (n-1)^2\left(\frac{e}{((n-1)\,b+e)\,(b-e)}\right) Q = \sum_{i=1}^{n} p_i.$$

Algebraic rearrangement yields

$$(n-1)\,Q = na - (b-e)\sum_{i=1}^{n} p_i.$$

Note that $\sum_{j\neq i} q_j = Q - q_i$. Then the consumer's first-order condition with respect to q_i can be written

$$\frac{a}{b-e} - (n-1)^2\left(\frac{1}{(n-1)\,b+e}\right) q_i - (n-1)\left(\frac{e}{((n-1)\,b+e)\,(b-e)}\right) Q = p_i.$$

This implies

$$q_i = \left(\frac{1}{n-1}\right)^2\left(\left(\frac{(n-1)\,b+e}{b-e}\right) a - \frac{e}{b-e}\left(na - (b-e)\sum_{j=1}^{n} p_j\right) - ((n-1)\,b+e)\,p_i\right)$$
$$= \left(\frac{1}{n-1}\right)^2\left((n-1)\,a - (n-1)\,bp_i + e\sum_{j\neq i} p_j\right).$$

This is the demand form given in (1). $\square$

Proof of Lemma 6. From Lemma 14, utility is given by (12). Substituting in demand and applying equilibrium symmetry,

$$\begin{aligned}\mathbb{E}\left[u\left(\mathbf{q}_t;\mathbf{p}_t\right)\right] &= \frac{2a}{b-e}\mathbb{E}\left[q_{it}\right] - \left(\frac{b}{b^2-e^2}\right)\mathbb{E}\left[q_{it}^2\right] - \left(\frac{e}{b^2-e^2}\right)\mathbb{E}\left[q_{it}q_{jt}\right] - 2\mathbb{E}\left[p_{it}q_{it}\right] \\ &= -2a\mathbb{E}\left[p_{it}\right] + b\mathbb{E}\left[p_{it}^2\right] - e\mathbb{E}\left[p_{it}p_{jt}\right].\end{aligned}$$

Expressing in terms of the expectation, covariance, and variance of prices,

$$\mathbb{E}\left[u\left(\mathbf{p}_t\right)\right] = \left(-2a + (b-e)\,\mathbb{E}\left[p_{it}\right]\right)\mathbb{E}\left[p_{it}\right] + b\operatorname{Var}\left(p_{it}\right) - e\operatorname{Cov}\left(p_{it},p_{jt}\right).$$

□

Proof of Lemma 7. In period t firm i's expected profits are

$$\mathbb{E}\left[\pi_{it}\right] = \mathbb{E}\left[\left(a - bp_{it} + ep_{jt}\right)\left(p_{it} - c_i\right)\right].$$

Note that, when considering ex ante expected profits, it is not necessary to condition on learned information, which disappears by the law of iterated expectations. Then we see

$$\begin{aligned}\mathbb{E}\left[\pi_{it}\right] &= a\mathbb{E}\left[p_{it}-c_i\right] - b\mathbb{E}\left[p_{it}^2 - p_{it}c_i\right] + e\mathbb{E}\left[p_{jt}p_{it} - p_{jt}c_i\right] \\ &= a\mathbb{E}\left[p_{it}-c_i\right] - b\operatorname{Var}\left(p_{it}\right) - b\mathbb{E}\left[p_{it}\right]^2 + b\operatorname{Cov}\left(p_{it},c_i\right) + b\mathbb{E}\left[p_{it}\right]\mathbb{E}\left[c_i\right] \\ &\qquad + e\operatorname{Cov}\left(p_{it},p_{jt}\right) + e\mathbb{E}\left[p_{it}\right]\mathbb{E}\left[p_{jt}\right] - e\operatorname{Cov}\left(p_{jt},c_i\right) - e\mathbb{E}\left[p_{jt}\right]\mathbb{E}\left[c_i\right] \\ &= \left(a - b\mathbb{E}\left[p_{it}\right] + e\mathbb{E}\left[p_{jt}\right]\right)\left(\mathbb{E}\left[p_{it}\right] - \mathbb{E}\left[c_i\right]\right) \\ &\qquad - b\left(\operatorname{Var}\left(p_{it}\right) - \operatorname{Cov}\left(p_{it},c_i\right)\right) + e\left(\operatorname{Cov}\left(p_{it},p_{jt}\right) - \operatorname{Cov}\left(p_{jt},c_i\right)\right) \\ &= \left(a - (b-e)\,\mathbb{E}\left[p_{it}\right]\right)\left(\mathbb{E}\left[p_{it}\right] - \mathbb{E}\left[c_i\right]\right) \\ &\qquad - b\left(\operatorname{Var}\left(p_{it}\right) - \operatorname{Cov}\left(p_{it},c_i\right)\right) + e\left(\operatorname{Cov}\left(p_{it},p_{jt}\right) - \operatorname{Cov}\left(p_{jt},c_i\right)\right).\end{aligned}$$

The final line follows by equilibrium symmetry. Since there are two firms, symmetry further implies that expected producer surplus is twice this quantity. □

Lemma 15. *For given values of b and e, consumer surplus decreases with information sharing when a is sufficiently large, provided $e \neq 0$.*

Proof. From Lemma 6 consumer surplus is

$$\mathbb{E}\left[u\left(\mathbf{p}\right)\right] = \left(-2a + (b-e)\mathbb{E}[p_i]\right)\mathbb{E}[p_i] + b\operatorname{Var}(p_i) - e\operatorname{Cov}(p_i,p_j)$$

Without information sharing, the expected price, variance of price, and covariance of prices in the first period are given by:[41]

$$
\begin{aligned}
\mathbb{E}\left[p_{i1}^{\star}\right] &= \frac{1}{2b-e}\left(a+b(\mu_\rho+\mu_\theta)+\frac{e^2\kappa^{\star}}{2(4b^2-e^2)}\mathbb{E}\left[\left(a-bc_i+e\mathbb{E}\left[\left.p_{j2}^{\star}\right|\rho,s_{i\rho},p_{j1}\right]\right)\right]\right);\\
\operatorname{Var}\left(p_{i1}^{\star}\right) &= \left[p_\theta^{\star}\right]^2\operatorname{Var}\left(\mathbb{E}\left[\theta_i|\,s_{i\theta}\right]\right)+\left[p_\rho^{\star}\right]^2\operatorname{Var}\left(\mathbb{E}\left[\rho|\,s_{i\rho}\right]\right)=\left[p_\theta^{\star}\right]^2\frac{\overline{\tau}_{i\theta}}{\tau_\theta}+\left[p_\rho^{\star}\right]^2\frac{\overline{\tau}_{i\rho}}{\tau_\rho};\\
\operatorname{Cov}\left(p_{i1}^{\star},p_{j1}^{\star}\right) &= \left[p_\rho^{\star}\right]^2\operatorname{Cov}\left(\mathbb{E}\left[\rho|\,s_{i\rho}\right],\mathbb{E}\left[\rho|\,s_{j\rho}\right]\right)=\left[p_\rho^{\star}\right]^2\frac{\overline{\tau}_{i\rho}^2}{\tau_\rho}.
\end{aligned}
$$

With information sharing these become

$$
\begin{aligned}
\mathbb{E}\left[p_{i1}^{c}\right] &= \frac{1}{2b-e}\left(a+b(\mu_\rho+\mu_\theta)+\frac{e^2\kappa^{c}}{2(4b^2-e^2)}\mathbb{E}\left[\left(a-bc_i+e\mathbb{E}\left[\left.p_{j2}^{c}\right|\rho,s_{\rho},p_{j1}\right]\right)\right]\right);\\
\operatorname{Var}\left(p_{i1}^{c}\right) &= \left[p_\theta^{c}\right]^2\operatorname{Var}\left(\mathbb{E}\left[\theta_i|\,s_{i\theta}\right]\right)+\left[p_\rho^{c}\right]^2\operatorname{Var}\left(\mathbb{E}\left[\rho|\,s_{\rho}\right]\right)=\left[p_\theta^{c}\right]^2\frac{\overline{\tau}_{i\theta}}{\tau_\theta}+\left[p_\rho^{c}\right]^2\frac{2\tau_{i\rho}}{\tau_\rho\left(\tau_\rho+2\tau_{i\rho}\right)};\\
\operatorname{Cov}\left(p_{i1}^{c},p_{j1}^{c}\right) &= \left[p_\rho^{c}\right]^2\operatorname{Var}\left(\mathbb{E}\left[\rho|\,s_{\rho}\right]\right)=\left[p_\rho^{c}\right]^2\frac{2\tau_{i\rho}}{\tau_\rho\left(\tau_\rho+2\tau_{i\rho}\right)}.
\end{aligned}
$$

Given an information sharing agreement the differences between the values with and without information sharing are

$$
\begin{aligned}
\Delta\mathbb{E}\left[p_{i1}\right] &= \left(\frac{1}{2b-e}\right)^2\left(\frac{e^2}{2(4b^2-e^2)}\right)\left(\kappa^c-\kappa^{\star}\right)\left(2a-2\left(b-e\right)\left(\mu_\rho+\mu_\theta\right)\right)b;\\
\Delta\operatorname{Var}\left(p_{i1}\right) &= \frac{\overline{\tau}_{i\theta}}{\tau_\theta}\left(\left[p_\theta^{c}\right]^2-\left[p_\theta^{\star}\right]^2\right)+\left[p_\rho^{c}\right]^2\frac{2\tau_{i\rho}}{\tau_\rho\left(\tau_\rho+2\tau_{s\rho i}\right)}-\left[p_\rho^{\star}\right]^2\frac{\overline{\tau}_{i\rho}}{\tau_\rho};\\
\Delta\operatorname{Cov}\left(p_{i1},p_{j1}\right) &= \left[p_\rho^{c}\right]^2\frac{2\tau_{i\rho}}{\tau_\rho\left(\tau_\rho+2\tau_{i\rho}\right)}-\left[p_\rho^{\star}\right]^2\frac{\overline{\tau}_{i\rho}^2}{\tau_\rho}.
\end{aligned}
$$

From Proposition 3 we have $\kappa^c > \kappa^{\star}$ for $e \neq 0$, so $\Delta\mathbb{E}[p_{i1}]$ is increasing in a; when expected demand is positive, so that $a > (b-e)\mathbb{E}[p]$, $\Delta\mathbb{E}[p_{i1}] > 0$. Equilibrium price coefficients (other than p_0) and price informativeness do not depend on the value of a, therefore $\Delta\operatorname{Var}(p_{i1})$ and $\Delta\operatorname{Cov}(p_{i1}, p_{j1})$ are constant for all a. Then the difference in consumer surplus across informational regimes depends only on the leading term in the expression for consumer

[41] In the variance and covariance given in this block and the subsequent, it is intuitive that price coefficients should be squared. Note also that, e.g., $\operatorname{Var}(\mathbb{E}[\theta_i|s_i]) = \operatorname{Var}(\overline{\tau}_{i\theta}s_{i\theta}) = \overline{\tau}_{i\theta}^2(\sigma_\theta^2+\sigma_{i\theta}^2) = \overline{\tau}_{i\theta}/\tau_\theta$, so the $\overline{\tau}_\theta$ term enters linearly.

surplus given in Lemma 6. The effect of a on this term is given by

$$
\begin{aligned}
&\frac{\partial}{\partial a}\left[\left(-2a+(b-e)\,\mathbb{E}\left[p_{i1}^{c}\right]\right)\mathbb{E}\left[p_{i1}^{c}\right]-\left(-2a+(b-e)\,\mathbb{E}\left[p_{i1}^{\star}\right]\right)\mathbb{E}\left[p_{i1}^{\star}\right]\right]\\
&=\frac{\partial}{\partial a}\left[-2a\Delta\mathbb{E}\left[p_{i1}\right]+(b-e)\,\Delta\mathbb{E}\left[p_{i1}\right]\left(\mathbb{E}\left[p_{i1}^{c}\right]+\mathbb{E}\left[p_{i1}^{\star}\right]\right)\right]\\
&=-2\Delta\mathbb{E}\left[p_{i1}\right]-2a\frac{\partial\Delta\mathbb{E}\left[p_{i1}\right]}{\partial a}\\
&\qquad+(b-e)\,\frac{\partial\Delta\mathbb{E}\left[p_{i1}\right]}{\partial a}\left(\mathbb{E}\left[p_{i1}^{c}\right]+\mathbb{E}\left[p_{i1}^{\star}\right]\right)+(b-e)\,\Delta\mathbb{E}\left[p_{i1}\right]\left(\frac{\partial\mathbb{E}\left[p_{i1}^{c}\right]}{\partial a}+\frac{\partial\mathbb{E}\left[p_{i1}^{\star}\right]}{\partial a}\right).
\end{aligned}
\tag{13}
$$

Since $e\neq 0$ implies $\kappa^{c}>\kappa^{\star}$, it follows that

$$
\begin{aligned}
\frac{\partial\mathbb{E}\left[p_{i1}^{c}\right]}{\partial a}&=\left(\frac{1}{2b-e}\right)\left(1+\left(\frac{e}{2b-e}\right)\frac{be}{4b^{2}-e^{2}}\kappa^{c}\right)\\
&>\left(\frac{1}{2b-e}\right)\left(1+\left(\frac{e}{2b-e}\right)\frac{be}{4b^{2}-e^{2}}\kappa^{\star}\right)=\frac{\partial\mathbb{E}\left[p_{i1}^{\star}\right]}{\partial a}.
\end{aligned}
$$

Then $\Delta\mathbb{E}[p_{i1}]>0$ allows (13) to be bounded above by

$$
\begin{aligned}
&\frac{\partial}{\partial a}\left[\left(-2a+(b-e)\,\mathbb{E}\left[p_{i1}^{c}\right]\right)\mathbb{E}\left[p_{i1}^{c}\right]-\left(-2a+(b-e)\,\mathbb{E}\left[p_{i1}^{\star}\right]\right)\mathbb{E}\left[p_{i1}^{\star}\right]\right]\\
&<\left(-2+2\,(b-e)\,\frac{\partial\mathbb{E}\left[p_{i1}^{c}\right]}{\partial a}\right)\Delta\mathbb{E}\left[p_{i1}\right]+\left(-2a+2\,(b-e)\,\mathbb{E}\left[p_{i1}^{c}\right]\right)\frac{\partial\Delta\mathbb{E}\left[p_{i1}\right]}{\partial a}.
\end{aligned}
$$

Straightforward algebraic rearrangment gives $-2+2(b-e)\partial\mathbb{E}[p_{i1}^{c}]/\partial a<0$, and by assumption $-2a+2(b-e)\mathbb{E}[p_{i1}^{c}]<0$. Since $\Delta\mathbb{E}[p_{i1}]$ is linear in a and $\partial\Delta\mathbb{E}[p_{i1}]/\partial a$ is constant in a, it follows that for any $\lambda>0$ there is $\underline{a}$ such that for all $a>\underline{a}$,

$$
\frac{\partial}{\partial a}\left[\left(-2a+(b-e)\,\mathbb{E}\left[p_{i1}^{c}\right]\right)\mathbb{E}\left[p_{i1}^{c}\right]-\left(-2a+(b-e)\,\mathbb{E}\left[p_{i1}^{\star}\right]\right)\mathbb{E}\left[p_{i1}^{\star}\right]\right]<-\lambda.
$$

Therefore we can choose an a such that

$$
\begin{aligned}
&\left(-2a+(b-e)\,\mathbb{E}\left[p_{i1}^{c}\right]\right)\mathbb{E}\left[p_{i1}^{c}\right]-\left(-2a+(b-e)\,\mathbb{E}\left[p_{i1}^{\star}\right]\right)\mathbb{E}\left[p_{i1}^{\star}\right]\\
&\quad<b\Delta\operatorname{Var}\left(p_{i1}\right)-e\Delta\operatorname{Cov}\left(p_{i1},p_{j1}\right).
\end{aligned}
\tag{14}
$$

For all such a, the consumer surplus in the first period decreases under information sharing.

We now consider the effect of information sharing on second period consumer welfare. In the proof of Theorem 3 it is shown that expected second period prices are unaffected by information sharing, so (following Lemma 6) consumer surplus will be completely determined by the effect of information sharing on the variance and covariance of prices.

The variance and covariance of second period prices are

$$
\begin{aligned}
\mathrm{Var}(p_{i2}^\star) &= \frac{1}{4}\mathrm{Var}(c_i) + \frac{e^2}{4b^2}\mathrm{Var}\left(\mathbb{E}[p_{j2}^\star|\rho, p_{i1}, p_{j1}]\right) - \frac{e}{4b}\mathrm{Cov}\left(c_i, \mathbb{E}[p_{j2}^\star|\rho, p_{i1}, p_{j1}]\right) \\
&= C_{\mathrm{Var}}\left(b,e\right)\sigma_\rho^2 + \frac{1}{4}\sigma_\theta^2 - \frac{e^2}{4(4b^2-e^2)}\kappa p_\theta^\star \mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right) \\
&\quad + \frac{4b^2e^2+e^4}{4(4b^2-e^2)^2}\left(\kappa^2\bar{\tau}_{i\rho}(1-\bar{\tau}_{i\rho})\sigma_\rho^2 p_\rho^{\star 2} + \kappa^2 p_\theta^{\star 2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)\right) \\
&= C_{\mathrm{Var}}\left(b,e\right)\sigma_\rho^2 + \frac{1}{4}\sigma_\theta^2 + \frac{e^4}{2\left(4b^2-e^2\right)^2}\kappa p_\theta^\star\, \mathrm{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right); \\
\mathrm{Cov}(p_{i2}^\star, p_{j2}^\star) &= \frac{1}{4}\mathrm{Cov}(c_i, c_j) + \frac{e}{2b}\mathrm{Cov}\left(c_i, \mathbb{E}[p_{j2}^\star|\rho, p_{i1}, p_{j1}]\right) \\
&\quad + \frac{e^2}{4b^2}\mathrm{Cov}\left(\mathbb{E}[p_{i2}^\star|\rho, p_{i1}, p_{j1}], \mathbb{E}[p_{j2}^\star|\rho, p_{i1}, p_{j1}]\right) \\
&= \frac{(2b+e)^2+4b^2e^2+e^4+4be^3}{4(4b^2-e^2)^2}\sigma_\rho^2 + \frac{e^2}{2(4b^2-e^2)}\kappa p_\theta^\star \mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right) \\
&\quad + \frac{be^3}{(4b^2-e^2)^2}\left(\kappa^2\bar{\tau}_{i\rho}(1-\bar{\tau}_{i\rho})\sigma_\rho^2 p_\rho^{\star 2} + \kappa^2 p_\theta^{\star 2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)\right) \\
&= C_{\mathrm{Cov}}\left(b,e\right) + \frac{4b^2e^2+2be^3-e^4}{2\left(4b^2-e^2\right)^2}\kappa p_\theta^\star\, \mathrm{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right).
\end{aligned}
$$

The above chains of equality follow from the fact that

$$
\kappa = \frac{p_\theta^\star \mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)}{\bar{\tau}_{i\rho}(1-\bar{\tau}_{i\rho})\sigma_\rho^2 p_\rho^{\star 2} + p_\theta^{\star 2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)}.
$$

Under information sharing the variance and covariance of prices are

$$
\begin{aligned}
\mathrm{Var}(p_{i2}^c) &= \frac{1}{4}\mathrm{Var}(c_i) + \frac{e^2}{4b^2}\mathrm{Var}\left(\mathbb{E}[p_{j2}^c|\rho, p_{i1}, p_{j1}]\right) - \frac{e}{4b}\mathrm{Cov}\left(c_i, \mathbb{E}[p_{j2}^c|\rho, p_{i1}, p_{j1}]\right) \\
&= C_{\mathrm{Var}}\left(b,e\right)\sigma_\rho^2 + \frac{1}{4}\sigma_\theta^2 + \frac{e^4}{2(4b^2-e^2)^2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right); \\
\mathrm{Cov}(p_{i2}^c, p_{j2}^c) &= \frac{1}{4}\mathrm{Cov}(c_i, c_j) + \frac{e}{2b}\mathrm{Cov}\left(c_i, \mathbb{E}[p_{j2}^c|\rho, p_{i1}, p_{j1}]\right) \\
&\quad + \frac{e^2}{4b^2}\mathrm{Cov}\left(\mathbb{E}[p_{i2}^c|\rho, p_{i1}, p_{j1}], \mathbb{E}[p_{j2}^c|\rho, p_{i1}, p_{j1}]\right) \\
&= \frac{(2b+e)^2+4b^2e^2+e^4+4be^3}{4(4b^2-e^2)^2}\sigma_\rho^2 + \frac{4b^2e^2-e^4+2be^3}{2(4b^2-e^2)^2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right).
\end{aligned}
$$

These terms are unaffected by the demand intercept a, and therefore (appealing to the arguments preceding equation (14)) information sharing decreases consumer surplus whenever a is sufficiently large. $\square$

Lemma 16. *When $e \approx b$, consumer surplus decreases with information sharing.*

Proof. We show that consumer surplus decreases with information sharing when $e = b$. Since all equilibrium expressions are continuous in the demand parameters a, b, and e, it follows that the same is true when $e \lesssim b$.

When $e = b$, Lemma 6 gives the difference in first period consumer surplus between informational regimes as

$$\begin{aligned}
\Delta \mathbb{E}[u] &= -2a\Delta \mathbb{E}[p_{i1}] + b\Delta \operatorname{Var}(p_{i1}) - e\Delta \operatorname{Cov}(p_{i1}, p_{j1}) \\
&= -2a\Delta \mathbb{E}[p_{i1}] + (\Delta \operatorname{Var}(p_{i1}) - \Delta \operatorname{Cov}(p_{i1}, p_{j1}))\, b.
\end{aligned}$$

From Theorem 3, $\Delta \mathbb{E}[p_{i1}] > 0$. The expressions in the proof of Lemma 15 imply that $\Delta \operatorname{Var}(p_{i1}) < \Delta \operatorname{Cov}(p_{i1}, p_{j1})$. The result follows immediately.

Regarding second period consumer surplus, we have

$$\begin{aligned}
\Delta \operatorname{Var}(p_{i2}) &= (1 - \kappa^\star p_\theta^\star) \left[\frac{e^4}{2(4b^2 - e^2)^2}\right] \operatorname{Var}(\mathbb{E}[\theta_i | s_{i\theta}]) \\
&= \frac{1}{4}(1 - \kappa^\star p_\theta^\star)\, \beta^2 \operatorname{Var}(\mathbb{E}[\theta_i | s_{i\theta}]), \text{ and} \\
\Delta \operatorname{Cov}(p_{i2}, p_{j2}) &= (1 - \kappa^\star p_\theta^\star) \left[\frac{4b^2e^2 + 2be^3 - e^4}{2(4b^2 - e^2)^2}\right] \operatorname{Var}(\mathbb{E}[\theta_i | s_{i\theta}]) \\
&= \frac{1}{2}(1 - \kappa^\star p_\theta^\star) \left[\frac{4b^2 + 2be - e^2}{4b^2 - e^2}\right] \beta \operatorname{Var}(\mathbb{E}[\theta_i | s_{i\theta}]).
\end{aligned}$$

Recalling that $\kappa^\star p_\theta^\star \leq 1$ (with equality only when $e = 0$), we check

$$\begin{aligned}
& b\Delta \operatorname{Var}(p_{i2}) - e\Delta \operatorname{Var}(p_{i2}, p_{j2}) \\
&= \frac{1}{2}(1 - \kappa^\star p_\theta^\star)\, b\beta^2 \operatorname{Var}(\mathbb{E}[\theta_i | s_{i\theta}]) - \frac{1}{2}(1 - \kappa^\star p_\theta^\star) \left[\frac{4b^2 + 2be - e^2}{4b^2 - e^2}\right] e\beta \operatorname{Var}(\mathbb{E}[\theta_i | s_{i\theta}]) \\
&\overset{\text{sign}}{=} \frac{r^2}{4 - r^2} - r\left(\frac{4 + 2r - r^2}{4 - r^2}\right) \overset{\text{sign}}{=} r - \left(4 + 2r - r^2\right) = r^2 - r - 4.
\end{aligned}$$

The roots of this quadratic are $\hat{r} \in [1 \pm \sqrt{17}]/2 \notin [-1, 1]$; then the above is negative for all valid r, and second period consumer surplus decreases with information sharing. $\square$

Lemma 17. *The effect of information sharing on expected second period producer surplus is*

$$\Delta \mathbb{E}[\Pi_2] = \left[\frac{-4 + 4r + 2r^2 - r^3}{4 - r^2}\right] (1 - \kappa^\star p_\theta^\star)\, b\beta \overline{\tau}_\theta \sigma_\theta^2.$$

Proof. Lemma 2 gives an expression for expected second period profits in terms of the expectation, variance, and covariance of prices. The proof of Theorem 3 shows that expected second period prices are unaffected by information sharing, and the proof of Lemma 16 gives expressions for the changes in variance and covariance of prices induced by information sharing. What remains is to consider the effect of information sharing on the covariance of prices and costs. We compute

$$\begin{aligned}
\operatorname{Cov}(c_i, p_{i2}) &= \operatorname{Cov}\left(c_i, \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p_{j2}\middle|\, \rho, \mathbf{p}_1\right]\right)\right) \\
&= \frac{1}{2}\left(\sigma_\theta^2 + \sigma_\rho^2\right) + \frac{1}{2} r \operatorname{Cov}\left(c_i, \frac{1}{4b^2 - e^2}\left((2b+e)\,a + 2b^2\mathbb{E}\left[c_j\middle|\, \rho, p_{j1}\right] + be\mathbb{E}\left[c_i\middle|\, \rho, p_{i1}\right]\right)\right) \\
&= \frac{1}{2}\left(\sigma_\theta^2 + \sigma_\rho^2\right) + \frac{1}{2}\left[\frac{e}{4b^2 - e^2}\right]\left(2b \operatorname{Cov}\left(c_i, \mathbb{E}\left[c_j\middle|\, \rho, p_{j1}\right]\right) + e \operatorname{Cov}\left(c_i, \mathbb{E}\left[c_i\middle|\, \rho, p_{i1}\right]\right)\right); \\
\operatorname{Cov}(c_i, p_{j2}) &= \operatorname{Cov}\left(c_i, \frac{1}{2b}\left(a + bc_j + e\mathbb{E}\left[p_{i2}\middle|\, \rho, \mathbf{p}_1\right]\right)\right) \\
&= \frac{1}{2}\sigma_\rho^2 + \frac{1}{2}\left[\frac{e}{4b^2 - e^2}\right]\left(2b \operatorname{Cov}\left(c_i, \mathbb{E}\left[c_i\middle|\, \rho, p_{i1}\right]\right) + e \operatorname{Cov}\left(c_i, \mathbb{E}\left[c_j\middle|\, \rho, p_{j1}\right]\right)\right).
\end{aligned}$$

Following Lemma 9, we compute

$$\begin{aligned}
\operatorname{Cov}\left(c_i, \mathbb{E}\left[c_i\middle|\, \rho, p_{i1}\right]\right) &= \operatorname{Cov}\left(c_i, \left(1 - \kappa\overline{\tau}_{i\rho}p_\rho\right)\rho + \kappa p_{i1}\right) \\
&= \left(1 - \kappa\overline{\tau}_{i\rho}p_\rho\right)\sigma_\rho^2 + \kappa \operatorname{Cov}\left(c_i, p_\theta\mathbb{E}\left[\theta_i\middle|\, s_i\right] + p_\rho\mathbb{E}\left[\rho\middle|\, s_i\right]\right) \\
&= \left(1 - \kappa\overline{\tau}_{i\rho}p_\rho\right)\sigma_\rho^2 + \kappa\overline{\tau}_{i\theta}p_\theta\sigma_\theta^2 + \kappa\overline{\tau}_{i\rho}p_\rho\sigma_\rho^2 \\
&= \sigma_\rho^2 + \kappa\overline{\tau}_{i\theta}p_\theta\sigma_\theta^2; \\
\operatorname{Cov}\left(c_i, \mathbb{E}\left[c_j\middle|\, \rho, p_{j1}\right]\right) &= \operatorname{Cov}\left(c_i, \left(1 - \kappa\overline{\tau}_{i\rho}p_\rho\right)\rho + \kappa p_{j1}\right) \\
&= \left(1 - \kappa\overline{\tau}_{i\rho}p_\rho\right)\sigma_\rho^2 + \kappa \operatorname{Cov}\left(c_i, p_\theta\mathbb{E}\left[\theta_j\middle|\, s_j\right] + p_\rho\mathbb{E}\left[\rho\middle|\, s_j\right]\right) \\
&= \left(1 - \kappa\overline{\tau}_{i\rho}p_\rho\right)\sigma_\rho^2 + \kappa\hat{\tau}_\rho p_\rho\sigma_\rho^2 = \sigma_\rho^2.
\end{aligned}$$

From this, it follows that

$$\begin{aligned}
\Delta \operatorname{Cov}(c_i, p_{i2}) &= \frac{1}{2}\beta\left(\kappa^c p_\theta^c\overline{\tau}_{i\theta}\sigma_\theta^2 - \kappa^\star p_\theta^\star\overline{\tau}_{i\theta}\sigma_\theta^2\right) = \frac{1}{2}\beta\left(1 - \kappa^\star p_\theta^\star\right)\operatorname{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right); \\
\Delta \operatorname{Cov}(c_i, p_{j2}) &= \frac{1}{r}\beta\left(1 - \kappa^\star p_\theta^\star\right)\operatorname{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right).
\end{aligned}$$

Following Lemma 7, this gives

$$
\begin{aligned}
\Delta\mathbb{E}\left[\Pi_2\right] &= 2b\left(\Delta\operatorname{Cov}\left(c_i, p_{i2}\right) - \Delta\operatorname{Var}\left(p_{i2}\right)\right) - 2e\left(\Delta\operatorname{Cov}\left(c_i, p_{j2}\right) - \Delta\operatorname{Cov}\left(p_{i2}, p_{j2}\right)\right) \\
&= 2b\left(\frac{1}{2}\beta\left(1-\kappa^\star p_\theta^\star\right)\operatorname{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right) - \frac{1}{2}\left(1-\kappa^\star p_\theta^\star\right)\beta^2\operatorname{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right)\right) \\
&\quad - 2e\left(\frac{1}{r}\beta\left(1-\kappa^\star p_\theta^\star\right) - \frac{1}{2}\left(1-\kappa^\star p_\theta^\star\right)\left[\frac{4b^2+2be-e^2}{4b^2-e^2}\right]\beta\right)\operatorname{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right) \\
&= \left(\left[b-b\beta\right] - \left[2b - \left(\frac{4b^2+2be-e^2}{4b^2-e^2}\right)e\right]\right)\left(1-\kappa^\star p_\theta^\star\right)\beta\operatorname{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right) \\
&= \left[\frac{-4b^3+be^2-be^2+4b^2e+2be^2-e^3}{4b^2-e^2}\right]\left(1-\kappa^\star p_\theta^\star\right)\beta\operatorname{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right) \\
&= \left[\frac{-4b^3+4b^2e+2be^2-e^3}{4b^2-e^2}\right]\left(1-\kappa^\star p_\theta^\star\right)\beta\operatorname{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right) \\
&= \left[\frac{-4+4r+2r^2-r^3}{4-r^2}\right]\left(1-\kappa^\star p_\theta^\star\right)b\beta\operatorname{Var}\left(\mathbb{E}\left[\theta_i|s_{i\theta}\right]\right).
\end{aligned}
$$

□

Corollary 5. *There is $\hat{r} \in (0,1)$ such that when $r > \hat{r}$, information sharing increases second period producer surplus, and when $r < \hat{r}$, information sharing decreases second period consumer producer surplus.*

Proof. This follows from a straightforward analysis of the expression in Lemma 17. □

Lemma 18. *The change in producer surplus induced by information sharing, $\Delta\mathbb{E}[\Pi_1]$, is increasing (decreasing) in the demand shifter a when $e > 0$ ($e < 0$). This increase (decrease) is at an increasing rate.*

Proof. The variance and covariance of equilibrium prices (and costs) are unaffected by elasticity a, and the expectation of second period prices is unaffected by information sharing, thus Lemma 7 gives that the effect of a on the effect of information sharing is determined by

$$
\Delta\mathbb{E}\left[\Pi_1\right] \simeq_a \left(a-(b-e)\,\mathbb{E}\left[c_i\right]\right)\Delta\mathbb{E}\left[p_{i1}\right] - (b-e)\left(\mathbb{E}\left[p_{i1}^c\right]^2 - \mathbb{E}\left[p_{i1}^\star\right]^2\right).
$$

Straightforward rearrangement of the equilibrium first order condition gives

$$
\mathbb{E}\left[p_{i1}\right] = \left(\frac{1}{2-r}\right)^2\left[\left(2-r+\hat{\kappa}\right)\frac{a}{b} + \left(2-r-\hat{\kappa}+r\hat{\kappa}\right)\mathbb{E}\left[c_i\right]\right].
$$

Here, $\hat{\kappa} \in \{\beta\kappa^c, \beta\kappa^\star\}$, where substituting in for $\hat{\kappa}$ gives the expected first period price in the corresponding informational regime. Thus,

$$\Delta\mathbb{E}\left[p_{i1}\right]=\left(\frac{1}{2-r}\right)^{2}\left(\frac{a}{b}-(1-r)\,\mathbb{E}\left[c_{i}\right]\right)\left(\kappa^{c}-\kappa^{\star}\right)\beta.$$

Additionally,

$$\frac{d\mathbb{E}\left[p_{i1}\right]^{2}}{da}=\frac{2}{b}\left(\frac{1}{2-r}\right)^{2}\left(2-r+\hat{\kappa}\right)\mathbb{E}\left[p_{i1}\right].$$

Then the change in the effect of information sharing on producer surplus is given by

$$\begin{aligned}
&\frac{d\Delta\mathbb{E}\left[\Pi_{1}\right]}{da}\\
&\propto b\left(\frac{1}{2-r}\right)^{2}\left(\frac{a}{b}-(1-r)\,\mathbb{E}\left[c_{i}\right]\right)\left(\kappa^{c}-\kappa^{\star}\right)+\left(a-(b-e)\,\mathbb{E}\left[c_{i}\right]\right)\left(\frac{1}{2-r}\right)^{2}\left(\kappa^{c}-\kappa^{\star}\right)\\
&\quad-2\,(b-e)\left(\frac{1}{2-r}\right)^{2}\left(\frac{2}{2-r}\left(\frac{a}{b}-(1-r)\,\mathbb{E}\left[c_{i}\right]\right)\left(\kappa^{c}-\kappa^{\star}\right)\right)\\
&\quad-2\,(b-e)\left(\frac{1}{2-r}\right)^{4}\left(\frac{a}{b}-(1-r)\,\mathbb{E}\left[c_{i}\right]\right)\left(\kappa^{c2}-\kappa^{\star 2}\right)\beta\\
&\overset{\text{sign}}{=}(2-r)^{2}\left(\frac{a}{b}-(1-r)\,\mathbb{E}\left[c_{i}\right]\right)-(1-r)\,(2-r)\left(\frac{a}{b}-(1-r)\,\mathbb{E}\left[c_{i}\right]\right)\\
&\quad-(1-r)\left(\frac{a}{b}-(1-r)\,\mathbb{E}\left[c_{i}\right]\right)\left(\kappa^{c}+\kappa^{\star}\right)\beta.
\end{aligned}$$

By assumption, $a\geq(b-e)\mathbb{E}[c_i]$, so we have

$$\frac{d\Delta\mathbb{E}\left[\Pi_{1}\right]}{da}\overset{\text{sign}}{=}(2-r)^{2}-2\,(1-r)\,(2-r)-(1-r)\left(\kappa^{c}+\kappa^{\star}\right)=(2-r)\,r-(1-r)\left(\kappa^{c}+\kappa^{\star}\right)\beta.$$

When $r<0$, this expression is clearly negative. When $r>0$, the sign depends on $\kappa^c+\kappa^\star$. We have $\beta\kappa^c=r^2/(2-r^2)\geq\beta\kappa^\star$. Then

$$\begin{aligned}
(2-r)\,r-(1-r)\left(\kappa^{c}+\kappa^{\star}\right)\beta &\geq (2-r)-(1-r)\,\frac{2r}{2-r^{2}}\\
&\overset{\text{sign}}{=}(2-r)\left(2-r^{2}\right)-2\,(1-r)\,r=4-4r+r^{3}>0.
\end{aligned}$$

Then when $r>0$, $d\Delta\mathbb{E}[\Pi_1]/da>0$. Lemma 17 shows that the effect of information sharing on second period profits does not depend on a. Because we have divided through by $a/b-(1-r)\mathbb{E}[c_i]>0$, which is linearly increasing in a, the result follows. □

Proof of Proposition 4. This follows immediately from Lemmas 15, 16, and 18. □

Lemma 19. *There exists a constant $C_u\in\mathbb{R}$ such that for any $\overline{\tau}_\rho$, the linear equilibrium* *with a large number of firms yields expected first period consumer surplus*

$$\mathbb{E}\left[u_{1\infty}\right] \propto (1-r)\operatorname{Var}\left(p_{i1}^{\star}\right) - r\operatorname{Cov}\left(p_{i1}^{\star}, p_{j1}^{\star}\right) + C_u.$$

Proof. For any finite number of firms n, the linear pricing equilibrium yields the first period expected consumer surplus given by

$$\begin{aligned} u\left(\mathbf{q_t};\mathbf{p_t}\right) = {} & \frac{a}{b-e}\sum_{i=1}^{n} q_{itn} - \frac{n-1}{2}\left(\frac{(n-1)\,b-(n-2)\,e}{((n-1)\,b+e)\,(b-e)}\right)\sum_{i=1}^{n} q_{itn}^{2} \\ & - \frac{n-1}{2}\left(\frac{e}{((n-1)\,b+e)\,(b-e)}\right)\sum_{i=1}^{n}\sum_{j\neq i} q_{itn}q_{jtn} - \sum_{i=1}^{n} p_{itn}q_{itn}. \end{aligned}$$

Let d_{in} be scaled demand with n firms,

$$d_{in} = (n-1)\,q_i = a - bp_{i1}^{\star} + \frac{e}{n-1}\sum_{j\neq i} p_{j1}^{\star}.$$

Applying symmetry of the linear pricing equilibrium and the linearity of expectation gives

$$\mathbb{E}\left[u_{1\infty}\right] = \lim_{n\nearrow\infty}\mathbb{E}\left[u_{1n}\right] \propto ab\mathbb{E}\left[d_{i\infty}\right] - \frac{1}{2}\,(b-e)\,\mathbb{E}\left[d_{i\infty}^{2}\right] - \frac{1}{2}e\mathbb{E}\left[d_{i\infty}d_{j\infty}\right] - (b-e)\,b\mathbb{E}\left[p_{i1}^{\star}d_{i\infty}\right].$$

In the limit with a large number of firms, $\mathbb{E}[p_{i1}^{\star}]$ does not depend on $\overline{\tau}_{\rho}$. Then $\mathbb{E}[d_{i\infty}]$ does not depend on $\overline{\tau}_{\rho}$. Define $d_{in}^{a} = d_{in} - a$. Then there is a constant $C_{\pi 1}$ such that the above equation can be written

$$\mathbb{E}\left[u_{1\infty}\right] \propto -(b-e)\,\mathbb{E}\left[\left(d_{i\infty}^{a} + 2bp_{i1}^{\star}\right)d_{i\infty}^{a}\right] - e\mathbb{E}\left[d_{i\infty}^{a}d_{j\infty}^{a}\right] + C_{\pi 1}.$$

We compute each piece in turn.

$$\begin{aligned}
\mathbb{E}\left[\left(d^a_{i\infty}+2bp^\star_{i1}\right)d^a_{i\infty}\right] &= \lim_{n\nearrow\infty}\mathbb{E}\left[\left(d^a_{in}+2bp^\star_{i1,n}\right)d^a_{in}\right]\\
&= \lim_{n\nearrow\infty}\mathbb{E}\left[-b^2p^{\star 2}_{i1,n}+\left(\frac{e}{n-1}\right)^2\left(\sum_{j\neq i}p^\star_{j1,n}\right)^2\right]\\
&= -b^2\mathbb{E}\left[p^{\star 2}_{i1}\right]+\lim_{n\nearrow\infty}\left(\frac{e}{n-1}\right)^2\mathbb{E}\left[\sum_{j\neq i}p^{\star 2}_{j1,n}+2\sum_{j\neq i}\sum_{k\neq i,j}p^\star_{j1,n}p^\star_{k1,n}\right]\\
&= -b^2\mathbb{E}\left[p^{\star 2}_{i1}\right]+2e^2\mathbb{E}\left[p^\star_{i1}p^\star_{j1}\right]=-b^2\operatorname{Var}\left(p^\star_{i1}\right)+2e^2\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)+C_{\pi 2};\\
\mathbb{E}\left[d^a_{i\infty}d^a_{j\infty}\right] &= \lim_{n\nearrow\infty}\mathbb{E}\left[d^a_{in}d^a_{jn}\right]\\
&= \lim_{n\nearrow\infty}\mathbb{E}\left[\left(-bp^\star_{i1,n}+\frac{e}{n-1}\sum_{k\neq i}p^\star_{k1,n}\right)\left(-bp^\star_{j1,n}+\frac{e}{n-1}\sum_{k\neq j}p^\star_{k1,n}\right)\right]\\
&= b^2\mathbb{E}\left[p^\star_{i1}p^\star_{j1}\right]-2be\mathbb{E}\left[p^\star_{i1}p^\star_{j1}\right]+\lim_{n\nearrow\infty}\left(\frac{e}{n-1}\right)^2\mathbb{E}\left[\left(\sum_{k\neq i,j}p^\star_{k1,n}\right)^2\right]\\
&= \left(b^2-2be\right)\mathbb{E}\left[p^\star_{i1}p^\star_{j1}\right]+2e^2\mathbb{E}\left[p^\star_{i1}p^\star_{j1}\right]=\left((b-e)^2+e^2\right)\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)+C_{\pi 3}.
\end{aligned}$$

Putting these pieces together leaves

$$\begin{aligned}
\mathbb{E}\left[u_{1\infty}\right] &\propto -(b-e)\left(-b^2\operatorname{Var}\left(p^\star_{i1}\right)+2e^2\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)\right)-e\left((b-e)^2+e^2\right)\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)+C_{\pi 4}\\
&= (b-e)\,b^2\operatorname{Var}\left(p^\star_{i1}\right)-\left(2e^2(b-e)+e(b-e)^2+e^3\right)\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)+C_{\pi 4}\\
&\propto (1-r)\operatorname{Var}\left(p^\star_{i1}\right)-r\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)+C_{\pi 5}.
\end{aligned}$$

□

Lemma 20. *There exists a constant $C_\pi\in\mathbb{R}$ such that for any $\overline{\tau}_\rho$, the linear equilibrium with a large number of firms yields expected first period producer surplus*

$$\mathbb{E}\left[\Pi_{1\infty}\right]\propto\left(\operatorname{Cov}\left(c_i,p^\star_{i1}\right)-\operatorname{Var}\left(p^\star_{i1}\right)\right)+\left(\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)-\operatorname{Cov}\left(c_i,p^\star_{j1}\right)\right)r+C_\pi.$$

Proof. For any finite number of firms n, the linear pricing equilibrium yields first period expected producer surplus of

$$\begin{aligned}\mathbb{E}\left[\Pi_{1n}\right] &= \mathbb{E}\left[\sum_{i=1}^{n} \frac{1}{n-1}\left(a - bp_{i1}^{\star} + \frac{e}{n-1}\sum_{j\neq i} p_{j1}^{\star}\right)\left(p_{i1}^{\star} - c_i\right)\right] \\ &= \mathbb{E}\left[\left(a - bp_{i1}^{\star} + \frac{e}{n-1}\sum_{j\neq i} p_{j1}^{\star}\right)\left(p_{i1}^{\star} - c_i\right)\right].\end{aligned}$$

Symmetry in the linear pricing equilibrium implies the second equality; this expression holds for any firm i. With the exception of sensitivity to opponent prices all terms are (first-order) independent of the number of firms n; prices themselves will depend on the number of the firms in the market. Linearity of expectations and symmetry of pricing strategies imply

$$\mathbb{E}\left[\Pi_{1\infty}\right] = \lim_{n\nearrow\infty} \mathbb{E}\left[\Pi_{1n}\right] = \mathbb{E}\left[\left(a - bp_{i1}^{\star}\right)\left(p_{i1}^{\star} - c_i\right)\right] + e\mathbb{E}\left[\left(p_{i1}^{\star} - c_i\right)p_{j1}^{\star}\right] \qquad (j \neq i).$$

Recall that when n is large, expected prices do not depend on $\overline{\tau}_{\rho}$. Then there are constants $C_{\pi k}$ such that

$$\begin{aligned}\mathbb{E}\left[\Pi_{1\infty}\right] &= e\mathbb{E}\left[\left(p_{i1}^{\star} - c_i\right)p_{j1}^{\star}\right] - b\mathbb{E}\left[\left(p_{i1}^{\star} - c_i\right)p_{i1}^{\star}\right] + C_{\pi 1} \\ &= e\mathbb{E}\left[p_{i1}^{\star}p_{j1}^{\star}\right] - e\mathbb{E}\left[c_i p_{j1}^{\star}\right] - b\mathbb{E}\left[p_{i1}^{\star 2}\right] + b\mathbb{E}\left[c_i p_{i1}^{\star}\right] + C_{\pi 1} \\ &= e\operatorname{Cov}\left(p_{i1}^{\star}, p_{j1}^{\star}\right) - e\operatorname{Cov}\left(c_i, p_{j1}^{\star}\right) - b\operatorname{Var}\left(p_{i1}^{\star}\right) + b\operatorname{Cov}\left(c_i, p_{i1}^{\star}\right) + C_{\pi 2} \\ &\propto \left(\operatorname{Cov}\left(c_i, p_{i1}^{\star}\right) - \operatorname{Var}\left(p_{i1}^{\star}\right)\right) + \left(\operatorname{Cov}\left(p_{i1}^{\star}, p_{j1}^{\star}\right) - \operatorname{Cov}\left(c_i, p_{i1}^{\star}\right)\right) r + C_{\pi 3}.\end{aligned}$$

This establishes the stated result. ☐

Proof of Lemma 8. This follows immediately from Lemmas 19 and 20. ☐

Proof of Proposition 5. Following Lemma 8, the extent to which producer surplus depends on $\overline{\tau}_{\rho}$ is given by

$$\mathbb{E}\left[\Pi_{1\infty}\right] \simeq \left(\operatorname{Cov}\left(c_i, p_{i1}^{\star}\right) - \operatorname{Var}\left(p_{i1}^{\star}\right)\right) + \left(\operatorname{Cov}\left(p_{i1}^{\star}, p_{j1}^{\star}\right) - \operatorname{Cov}\left(c_i, p_{j1}^{\star}\right)\right) r.$$

We compute in turn:

$$\begin{aligned}\operatorname{Cov}\left(c_i, p_{i1}^{\star}\right) &\simeq_{\overline{\tau}_{\rho}} p_{\rho\infty}\overline{\tau}_{\rho}\sigma_{\rho}^2; \\ \operatorname{Var}\left(p_{i1}^{\star}\right) &\simeq_{\overline{\tau}_{\rho}} p_{\rho\infty}^2\overline{\tau}_{\rho}^2\left(\sigma_{\rho}^2 + \sigma_{\varepsilon\rho}^2\right) = p_{\rho\infty}^2\overline{\tau}_{\rho}\sigma_{\rho}^2; \\ \operatorname{Cov}\left(p_{i1}^{\star}, p_{j1}^{\star}\right) &\simeq_{\overline{\tau}_{\rho}} p_{\rho\infty}^2\overline{\tau}_{\rho}^2\sigma_{\rho}^2; \\ \operatorname{Cov}\left(c_i, p_{j1}^{\star}\right) &\simeq_{\overline{\tau}_{\rho}} p_{\rho\infty}\overline{\tau}_{\rho}\sigma_{\rho}^2.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\left[\Pi_{1\infty}\right] &\simeq_{\overline{\tau}_\rho} \left(p_{\rho\infty}\overline{\tau}_\rho\sigma_\rho^2 - p_{\rho\infty}^2\overline{\tau}_\rho\sigma_\rho^2\right) + \left(p_{\rho\infty}^2\overline{\tau}_\rho^2\sigma_\rho^2 - p_{\rho\infty}\overline{\tau}_\rho\sigma_\rho^2\right) r \\ &\propto (1-r)\, p_{\rho\infty}\overline{\tau}_\rho - (1 - r\overline{\tau}_\rho)\, p_{\rho\infty}^2\overline{\tau}_\rho \\ &= \frac{(1-r)\,\overline{\tau}_\rho}{2 - r\overline{\tau}_\rho} - \frac{(1-r\overline{\tau}_\rho)\,\overline{\tau}_\rho}{(2-r\overline{\tau}_\rho)^2} = \frac{(1-2r)\,\overline{\tau}_\rho + r^2\overline{\tau}_\rho^2}{(2-r\overline{\tau}_\rho)^2}.\end{aligned}$$

Equilibrium expected profits with information sharing can be computed by setting $\overline{\tau}_\rho = 1$ in the above equation. Then determining whether information sharing improves producer surplus is equivalent to solving

$$\begin{aligned} & \frac{(1-2r)\,\overline{\tau}_\rho + r^2\overline{\tau}_\rho}{(2-r\overline{\tau}_\rho)^2} \gtrless \frac{(1-2r)+r^2}{(1-2r)^2} \\ \Longleftrightarrow \quad & (1-r)^2(2-r)^2\,\overline{\tau}_\rho - (1-\overline{\tau}_\rho)(2-r)^2 \gtrless (1-r)^2(2-r\overline{\tau}_\rho)^2 \\ \Longleftrightarrow \quad & (1-r)^2\left(-4(1-\overline{\tau}_\rho) + r^2\overline{\tau}_\rho(1-\overline{\tau}_\rho)\right) \gtrless (1-\overline{\tau}_\rho)(2-r)^2 \\ \Longleftrightarrow \quad & -(1-r)^2\left(4 - r^2\overline{\tau}_\rho\right) \gtrless (2-r)^2.\end{aligned}$$

Since the left-hand side is negative and the right-hand side is positive, the inequality realizes as $<$, and information sharing improves producer surplus. □

Proof of Proposition 6. Following Lemma 8, the extent to which consumer surplus depends on $\overline{\tau}_\rho$ is given by

$$\mathbb{E}\left[u_{1\infty}\middle|\,\overline{\tau}_\rho\right] \simeq_{\overline{\tau}_\rho} (1-r)\operatorname{Var}(p_{i1}^\star) - r\operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right).$$

We compute in turn:

$$\begin{aligned}\operatorname{Var}(p_{i1}^\star) &\simeq_{\overline{\tau}_\rho} p_{\rho\infty}^2\overline{\tau}_\rho\sigma_\rho^2; \\ \operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) &\simeq_{\overline{\tau}_\rho} p_{\rho\infty}^2\overline{\tau}_\rho^2\sigma_\rho^2.\end{aligned}$$

Then

$$\mathbb{E}\left[u_{1\infty}\middle|\,\overline{\tau}_\rho\right] \simeq_{\overline{\tau}_\rho} (1-r)\, p_{\rho\infty}^2\overline{\tau}_\rho\sigma_\rho^2 - r p_{\rho\infty}^2\overline{\tau}_\rho^2\sigma_\rho^2 \propto \frac{(1-r)\,\overline{\tau}_\rho - r\overline{\tau}_\rho^2}{(2-r\overline{\tau}_\rho)^2}.$$

To determine the effects of information sharing, we compare expected consumer surplus with precision $\overline{\tau}_\rho$ against precision $\overline{\tau}'_\rho = 1$,

$$
\begin{aligned}
& & \mathbb{E}\left[u_{1\infty}\middle|\,\overline{\tau}_\rho\right] \gtreqless \mathbb{E}\left[u_{1\infty}\middle|\,\overline{\tau}'_\rho = 1\right] \\
\iff & & \frac{(1-r)\,\overline{\tau}_\rho - r\overline{\tau}_\rho^2}{\left(2 - r\overline{\tau}_\rho\right)^2} \gtreqless \frac{1-2r}{\left(2-r\right)^2} \\
\iff & \left(r^3 + 3r^2 - 4r\right)\overline{\tau}_\rho^2 + \left(4 - 4r - r^3 - 3r^2\right)\overline{\tau}_\rho + (8r-4) \gtreqless 0. &
\end{aligned} \tag{15}
$$

When $\overline{\tau}_\rho = 0$ the left-hand side is $8r - 4$, and the value of information sharing will depend on whether $r \gtreqless 1/2$. When $\overline{\tau}_\rho = 1$, the left-hand side is 0. The left-hand side of the above inequality is a quadratic in $\overline{\tau}_\rho$, so properties of the parabola will determine the effect of information sharing on consumer surplus. In particular, it is sufficient to analyze the slope of the parabola at $\overline{\tau}_\rho = 1$. If this slope is positive the left-hand side of (15) is negative and information sharing improves expected consumer welfare; if this slope is negative information sharing may lower expected consumer welfare, depending on initial informational precision.

The derivative of the left-hand side of (15) with respect to $\overline{\tau}_\rho$, evaluated at $\overline{\tau}_\rho = 1$, is

$$
2\left(r^3 + 3r^2 - 4r\right) + \left(4 - 4r - r^3 - 3r^2\right) = r^3 + 3r^2 - 12r + 4 = -\left(2-r\right)\left(r^2 + 5r - 2\right).
$$

Then the slope of the left-hand term will depend on $-(r^2+5r-2) \gtreqless 0$. Solving the quadratic gives

$$
r^{\perp} = -\frac{5}{2} + \frac{1}{2}\sqrt{33} \approx 0.372.
$$

When $r \lesssim 0.372$ the slope of the left-hand term is positive at $\overline{\tau}_\rho = 1$, and when $r \gtrsim 0.372$ the slope of the left-hand term is negative at $\overline{\tau}_\rho = 1$. Then when $r \lesssim 0.372$ information sharing strictly improves expected consumer surplus, and when $r \gtrsim 0.372$ information sharing may harm expected consumer surplus, depending on the initial level of precision $\overline{\tau}_\rho$.

Finally, the derivative of the left-hand side of (15) is $(4-6r-3r^2)(\overline{\tau}_\rho - \overline{\tau}_\rho^2) + 8 > 0$. Then the negative effect of information sharing on consumer welfare is increasing in substitutability. When $r = 1/2$ the left-hand side of (15) is $9(\overline{\tau}_\rho - \overline{\tau}_\rho^2)/8 \geq 0$, and hence for all $r \geq 1/2$ the left-hand side of (15) is weakly positive. Then for all $r \geq 1/2$ information sharing decreases consumer welfare, strictly so when either $\overline{\tau}_\rho \in (0, 1)$ or $r > 1/2$.

□

C Proofs for Section 5

Per footnote 35, in this appendix we shorten the number of signals firm i shares with firm j, $\tilde{M}_{i\to jx}$, to $\tilde{M}_{jx}$. This notation is less unwieldy and is consistent with the literature (see, e.g., Vives [2001]).

Proof of Theorem 4. This proof follows in a similar fashion to that of Lemmas 9 and 10.

The public information that is relevant to firm j's marginal cost includes, ρ, p_{j1}, shared information on ρ: $\tilde{s}_\rho = \rho + \frac{1}{\tilde{M}_{j\rho}+\tilde{M}_{i\rho}}\left(\sum_{m=1}^{\tilde{M}_{j\rho}} u_{i\rho m} + \sum_{m=1}^{\tilde{M}_{i\rho}} u_{j\rho m}\right)$, and the information shared with firm i about θ_j: $\tilde{s}_{i\theta_j}$.[42] Additionally, because ρ is observed, then the remainder of the first period public signal on common costs can be identified: $\tilde{\varepsilon}_\rho \equiv \tilde{s}_\rho - \rho$. Given a linear strategy in the first period, the five variables $(\theta_j, \rho, \hat{p}_{j1}, \tilde{\varepsilon}_\rho, \tilde{s}_{i\theta_j})^T$ are distributed joint normally with means $(\mu_\theta, \mu_\rho, \mathbb{E}\left[\hat{p}_{j1}\right], 0, \mu_\theta)^T$ and covariance matrix

$$\begin{pmatrix} \sigma_\theta^2 & 0 & \tilde{p}_{j\theta_j}\tilde{\bar{\tau}}_{j\theta_j}\sigma_\theta^2 & 0 & \sigma_\theta^2 \\ 0 & \sigma_\rho^2 & \tilde{p}_{j\rho}\tilde{\bar{\tau}}_{j\rho}\sigma_\rho^2 & 0 & 0 \\ \tilde{p}_{j\theta}\tilde{\bar{\tau}}_{j\theta_j}\sigma_\theta^2 & \tilde{p}_{j\rho}\tilde{\bar{\tau}}_{j\rho}\sigma_\rho^2 & \tilde{p}_{j\theta_j}^2\tilde{\bar{\tau}}_{j\theta_j}\sigma_\theta^2 + \tilde{p}_{j\rho}^2\tilde{\bar{\tau}}_{j\rho}\sigma_\rho^2 & p_{j\rho}\tilde{\bar{\tau}}_{j\rho}\left(\frac{\sigma_{u\rho}^2}{M_{j\rho}+\tilde{M}_{j\rho}}\right) & \tilde{p}_{j\theta_j}\tilde{\bar{\tau}}_{j\theta_j}\left(\sigma_\theta^2 + \frac{\sigma_{u\theta_j}}{M_{j\theta_j}}\right) \\ 0 & 0 & p_{j\rho}\tilde{\bar{\tau}}_{j\rho}\left(\frac{\sigma_{u\rho}^2}{M_{j\rho}+\tilde{M}_{j\rho}}\right) & \frac{\sigma_{u\rho}^2}{\tilde{M}_{j\rho}+\tilde{M}_{i\rho}} & 0 \\ \sigma_\theta^2 & 0 & \tilde{p}_{j\theta_j}\tilde{\bar{\tau}}_{j\theta_j}\left(\sigma_\theta^2 + \frac{\sigma_{u\theta_j}}{M_{j\theta_j}}\right) & 0 & \sigma_\theta^2 + \frac{\sigma_{u\theta_j}^2}{\tilde{M}_{i\theta_j}} \end{pmatrix}.$$

Then the conditional expectation of c_j, given ρ, $\hat{p}_{j1}$, $\tilde{\varepsilon}_\rho$, and $\tilde{s}_{i\theta_j}$ is

$$\mathbb{E}\left[\theta_j \middle| \rho, \hat{p}_{j1}, \tilde{\varepsilon}_\rho, \tilde{s}_{i\theta_j}\right] = \mu_\theta + \Sigma_{12}\Sigma_{22}^{-1}\left((\rho, \hat{p}_{j1}, \tilde{\varepsilon}_\rho, \tilde{s}_{i\theta_j})^T - (\mu_\rho, \mathbb{E}\left[\hat{p}_{j1}\right], 0, \mu_\theta)^T\right), \text{ with}$$

$$\Sigma_{12} = \left(0, \tilde{p}_{j\theta_j}\tilde{\bar{\tau}}_{j\theta_j}\sigma_\theta^2, 0, \sigma_\theta^2\right),$$

$$\Sigma_{22} = \begin{pmatrix} \sigma_\rho^2 & \tilde{p}_{j\rho}\tilde{\bar{\tau}}_{j\rho}\sigma_\rho^2 & 0 & 0 \\ \tilde{p}_{j\rho}\tilde{\bar{\tau}}_{j\rho}\sigma_\rho^2 & \tilde{p}_{j\theta_j}^2\tilde{\bar{\tau}}_{j\theta_j}\sigma_\theta^2 + \tilde{p}_{j\rho}^2\tilde{\bar{\tau}}_{j\rho}\sigma_\rho^2 & p_{j\rho}\tilde{\bar{\tau}}_{j\rho}\left(\frac{\sigma_{u\rho}^2}{M_{j\rho}+\tilde{M}_{j\rho}}\right) & \tilde{p}_{j\theta_j}\tilde{\bar{\tau}}_{j\theta_j}\left(\sigma_\theta^2 + \frac{\sigma_{u\theta_j}}{M_{j\theta_j}}\right) \\ 0 & p_{j\rho}\tilde{\bar{\tau}}_{j\rho}\left(\frac{\sigma_{u\rho}^2}{M_{j\rho}+\tilde{M}_{j\rho}}\right) & \frac{\sigma_{u\rho}^2}{\tilde{M}_{j\rho}+\tilde{M}_{i\rho}} & 0 \\ 0 & \tilde{p}_{j\theta_j}\tilde{\bar{\tau}}_{j\theta_j}\left(\sigma_\theta^2 + \frac{\sigma_{u\theta_j}}{M_{j\theta_j}}\right) & 0 & \sigma_\theta^2 + \frac{\sigma_{u\theta_j}^2}{\tilde{M}_{i\theta_j}} \end{pmatrix}.$$

The conditional expectation matrix is

$$\Sigma_{21}\Sigma_{22}^{-1} = \left(-\tilde{p}_{j\rho}\tilde{\bar{\tau}}_{j\rho}\tilde{\kappa},\ \tilde{\kappa},\ -\tilde{p}_{j\rho}\tilde{\bar{\tau}}_{j\rho}\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{j\rho}+\tilde{M}_{j\rho}}\tilde{\kappa},\ \tilde{\bar{\tau}}_{i\theta_j}(1-\tilde{\kappa}\tilde{p}_{j\theta_j})\right),$$

[42] Note that $\tilde{s}_{j\theta_i}$ is public information in period two that impacts p_{j1}. In equilibrium this is known and can be accounted for, and therefore this public signal has no influence on expected cost of firm j in the second period.

where

$$\tilde{\kappa} = \frac{\tilde{p}_{j\theta_j}\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2\left(1-\frac{\tilde{M}_{i\theta_j}}{M_{j\theta_j}}\frac{M_{j\theta_j}\sigma_\theta^2+\sigma_{u\theta_j}^2}{\tilde{M}_{i\theta_j}\sigma_\theta^2+\sigma_{u\theta_j}^2}\right)}{\tilde{p}_{j\theta_j}^2\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2\left(1-\frac{\tilde{M}_{i\theta_j}}{M_{j\theta_j}}\frac{M_{j\theta_j}\sigma_\theta^2+\sigma_{u\theta_j}^2}{\tilde{M}_{i\theta_j}\sigma_\theta^2+\sigma_{u\theta_j}^2}\right)+\tilde{p}_{j\rho}^2\tilde{\tilde{\tau}}_{j\rho}\left(1-\tilde{\tilde{\tau}}_{j\rho}\right)\sigma_\rho^2\left(1-\frac{\tilde{M}_{j\rho}+\tilde{M}_{i\rho}}{M_{j\rho}+\tilde{M}_{j\rho}}\right)}.$$

Then the expected costs as a function of the public information in the second period are given by

$$\begin{aligned}\mathbb{E}[c_j|\rho,\hat{p}_{j1},\tilde{\varepsilon}_\rho,\tilde{s}_{i\theta_j}] &= \rho+\mu_\theta-(\rho-\mu_\rho)\tilde{p}_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\tilde{\kappa}+\left(\hat{p}_{j1}-\mathbb{E}[\hat{p}_{j1}]\right)\tilde{\kappa}\\ &\quad-\tilde{\varepsilon}_\rho\tilde{p}_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{j\rho}+\tilde{M}_{j\rho}}\tilde{\kappa}+(1-\tilde{\kappa}\tilde{p}_{j\theta_j})\tilde{\tilde{\tau}}_{i\theta_j}(\tilde{s}_{i\theta_j}-\mu_\theta).\end{aligned}$$

Firm i's expectation of these expected costs given history, h_{i1}, are

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}\left[c_j|\rho,p_{j1},\tilde{\varepsilon}_\rho,\tilde{s}_{i\theta_j}\right]|h_{i1}\right]=\mathbb{E}[\rho|\tilde{s}_{i\rho}]+\mathbb{E}[\theta_j|\tilde{s}_{i\theta_j}],\\ &\mathbb{E}\left[\mathbb{E}\left[c_i|\rho,p_{j1},\tilde{\varepsilon}_\rho,\tilde{s}_{i\theta_j}\right]|h_{i1}\right]=\mathbb{E}[\rho|\tilde{s}_{i\rho}]+\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]+\tilde{\kappa}\tilde{p}_{i\theta_i}\left(\mathbb{E}[\theta_i|\tilde{s}_{i\theta_i}]-\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]\right)\\ &\qquad+\tilde{\kappa}\tilde{p}_{i\rho}\left(\mathbb{E}[\rho|\tilde{s}_{i\rho}]-\tilde{\tilde{\tau}}_{i\rho}\left(1-\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{\tilde{M}_{i\rho}+M_{i\rho}}\right)\mathbb{E}[\rho|\tilde{s}_{i\rho}]-\tilde{\tilde{\tau}}_{i\rho}\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{\tilde{M}_{i\rho}+M_{i\rho}}\tilde{s}_\rho-(1-\tilde{\tilde{\tau}}_{i\rho})\mu_\rho\right).\end{aligned}$$

Expected first period price of the firm j is

$$\begin{aligned}\mathbb{E}[\tilde{p}_{j1}|h_{i1}] &= \mathbb{E}\left[\tilde{p}_{j0}+\tilde{p}_{j\theta_j}\mathbb{E}[\theta_j|\tilde{s}_{j\theta_j}]+\tilde{p}_{j\rho}\mathbb{E}[\rho|\tilde{s}_{j\rho}]+\tilde{p}_{j\theta_i}\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]+\tilde{p}_{j\tilde{s}_{i\theta_j}}\mathbb{E}[\theta_j|\tilde{s}_{i\theta_j}]+\tilde{p}_{j\tilde{s}_\rho}\mathbb{E}[\rho|\tilde{s}_\rho]|h_{i1}\right]\\ &= \tilde{p}_{j0}+\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\tilde{p}_{j\rho}\mathbb{E}[\rho|\tilde{s}_\rho]+\left(1-\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)\left(\tilde{\tilde{\tau}}_{j\rho}\mathbb{E}[\rho|\tilde{s}_{i\rho}]+(1-\tilde{\tilde{\tau}}_{j\rho})\mu_\rho\right)\tilde{p}_{j\rho}\\ &\quad+\tilde{p}_{j\theta_j}\mathbb{E}[\theta_j|\tilde{s}_{i\theta_j}]+\tilde{p}_{j\tilde{s}_{i\theta_j}}\mathbb{E}[\theta_i|\tilde{s}_{i\theta_j}]+\tilde{p}_{j\tilde{s}_\rho}\mathbb{E}[\rho|\tilde{s}_\rho]+\tilde{p}_{j\theta_i}\mathbb{E}[\theta_i|\tilde{s}_{j\theta_1}].\end{aligned}$$

Plugging these into the first order condition from Lemma 3

$$\begin{aligned}&4b\tilde{p}_{i1}=2\mathbb{E}[bc_i+a+e\tilde{p}_{j1}|h_{i1}]+\beta\tilde{\kappa}_i\mathbb{E}\left[\left(a-bc_i+\frac{e}{4b^2-e^2}((2b+e)a)\right)\middle|h_{i1}\right]\\ &+\frac{e\beta\tilde{\kappa}_i}{4b^2-e^2}\left[2b^2\left(\mathbb{E}[\rho|\tilde{s}_{i\rho}]+\mathbb{E}[\theta_j|\tilde{s}_{i\theta_j}]\right)+be\left(\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]+\tilde{\kappa}_i\tilde{p}_{i\theta_i}\left(\mathbb{E}[\theta_i|\tilde{s}_{i\theta_i}]-\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]+\mathbb{E}[\rho|\tilde{s}_{i\rho}]\right)\right)\right]\\ &+\frac{be^2\beta\tilde{\kappa}_i}{4b^2-e^2}\left[\tilde{\kappa}\tilde{p}_{i\rho}\left(\left(1-\tilde{\tilde{\tau}}_{i\rho}\left(1-\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{\tilde{M}_{i\rho}+M_{i\rho}}\right)\right)\left(\mathbb{E}[\rho|\tilde{s}_{i\rho}]-\mu_\rho\right)-\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{\tilde{M}_{i\rho}+M_{i\rho}}\mathbb{E}[\rho|\tilde{s}_\rho]\right)\right].\end{aligned}$$

Matching coefficients we get the following system:

$$
\begin{aligned}
4bp_{i\theta_i} &= 2b - b\beta\tilde{\kappa}_i + \frac{be^2\beta\tilde{\kappa}_i^2 p_{i\theta}}{4b^2 - e^2}; \\
4bp_{i\rho} &= 2b - b\beta\tilde{\kappa}_i + \frac{be^2\beta\tilde{\kappa}_i}{4b^2 - e^2}\left(1 + \kappa_i p_{i\rho}\left(1 - \tilde{\tilde{\tau}}_\rho\left(1 - \frac{\tilde{M}_{i\rho} + \tilde{M}_{j\rho}}{M_{i\rho} + \tilde{M}_{i\rho}}\right)\right)\right) + \frac{2b^2 e\beta\tilde{\kappa}_i}{4b^2 - e^2} \\
&\quad + 2e\left(1 - \frac{\tilde{M}_{i\rho} + \tilde{M}_{j\rho}}{M_{i\rho} + \tilde{M}_{i\rho}}\right)\tilde{\tilde{\tau}}_\rho p_{j\rho}; \\
4bp_{i\theta_j} &= 2e\left(\tilde{p}_{j\theta_j} + \tilde{p}_{j\tilde{s}_{i\theta_j}}\right) + \frac{2b^2 e\beta\tilde{\kappa}_i}{4b^2 - e^2}; \\
4bp_{i\tilde{s}_{j\theta_i}} &= 2e\tilde{p}_{j\theta_i} + \frac{be^2\beta\tilde{\kappa}_i}{4b^2 - e^2}(1 - \tilde{\kappa}_i\tilde{p}_{i\theta_i}); \\
4b\tilde{p}_{i\tilde{s}_\rho} &= 2e\left(\tilde{p}_{j\tilde{s}_\rho} + \frac{\tilde{M}_{i\rho} + \tilde{M}_{j\rho}}{M_{i\rho} + \tilde{M}_{i\rho}}\tilde{p}_{j\rho}\right) - \frac{be^2\beta\tilde{\kappa}_i^2 p_{i\rho}}{4b^2 - e^2}\left(\frac{\tilde{M}_{i\rho} + \tilde{M}_{j\rho}}{M_{i\rho} + \tilde{M}_{i\rho}}\right).
\end{aligned}
$$

Imposing symmetry (in both strategies and shared information ($\tilde{M}_{i\rho} = \tilde{M}_{j\rho}$ and $\tilde{M}_{i\theta_j} = \tilde{M}_{j\theta_i}$)) and rearranging the coefficients we get the desired coefficients. Combining these with the equation for the informativeness of price we get the following expression for the equilibrium value of $\hat{\kappa} = \beta\tilde{\kappa}$

$$
\begin{aligned}
&\hat{\kappa}\left(1 - \left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2 (2+\hat{\kappa})^2 \tilde{\tilde{\tau}}_{i\rho}(1 - \tilde{\tilde{\tau}}_{i\rho})\sigma_\rho^2 \\
&= (\beta(2+\hat{\kappa}) - \hat{\kappa})\,\tilde{\tilde{\tau}}_{i\theta_i}\sigma_\theta^2\left(1 - \frac{\tilde{M}_{i\theta_j}}{M_{j\theta_j}}\frac{M_{j\theta_j}\sigma_\theta^2 + \sigma_{u\theta_j}^2}{\tilde{M}_{i\theta_j}\sigma_\theta^2 + \sigma_{u\theta_j}^2}\right) \\
&\quad \times \frac{\left(2 - \left(1 - \frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)\left(r\tilde{\tilde{\tau}}_{j\rho} - \frac{1}{2}\hat{\kappa}^2(1 - \tilde{\tilde{\tau}}_{j\rho})\right) - r\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)^2}{\left(1 - \frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)}
\end{aligned}
\tag{16}
$$

The argument of existence outlined in the proof of Theorem 1 applies to this more general setting. □

Proof of Proposition 7. Following the proof of Theorem 2 we calculate the ex-ante expected price in period 1 and 2.

Ex-ante expected first period prices are

$$
\begin{aligned}
\mathbb{E}[\tilde{p}_{i1}] &= \frac{1}{2b-e}\mathbb{E}\left[\mathbb{E}\left[bc_i + a + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[\tilde{p}_{j2}\middle|\,\rho, \hat{p}_{j1}, \tilde{\varepsilon}_\rho, \tilde{s}_{i\theta_j}\right]\right)\frac{be}{4b^2 - e^2}\tilde{\kappa}\middle|\, h_{i1}\right]\right] \\
&= \frac{1}{2b-e}\left(a + b(\mu_\rho + \mu_\theta) + \frac{e^2\tilde{\kappa}}{2(4b^2 - e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[\tilde{p}_{j2}\middle|\,\rho, \hat{p}_{j1}, \tilde{\varepsilon}_\rho, \tilde{s}_{i\theta_j}\right]\right)\middle|\, h_{i1}\right]\right]\right).
\end{aligned}
$$

Where ex-ante expected second period prices do not depend on the information sharing agreement:

$$\begin{aligned}
\mathbb{E}[\tilde{p}_{j2}] &= \frac{1}{4b^2-e^2}\left((2b+e)\,a + 2b^2\mathbb{E}\left[\mathbb{E}\left[\left.c_j\right|\rho,\hat{p}_{j1},\tilde{\tilde{\varepsilon}}_\rho,\tilde{s}_{i\theta_j}\right]\right] + be\mathbb{E}\left[\mathbb{E}\left[\left.c_i\right|\rho,\hat{p}_{i1},\tilde{\tilde{\varepsilon}}_\rho,\tilde{s}_{j\theta_i}\right]\right]\right)\\
&= \frac{1}{4b^2-e^2}\left(\mathbb{E}\left[(2b+e)a + 2b^2\left(\mathbb{E}\left[\theta_j|\tilde{s}_{i\theta_j}\right] + \mathbb{E}\left[\rho|\tilde{s}_{i\rho}\right]\right) + be\left(\mathbb{E}\left[\theta_i|\tilde{s}_{j\theta_i}\right] + \mathbb{E}\left[\rho|\tilde{s}_{i\rho}\right]\right)\right]\right)\\
&= \frac{a+b\mathbb{E}[c_i]}{2b-e}
\end{aligned}$$

Therefore ex-ante expected prices in the first period become

$$\begin{aligned}
\mathbb{E}[\tilde{p}_{i1}] &= \frac{a+b\mathbb{E}[c_i]}{2b-e} + \frac{e^2\tilde{\kappa}}{2(4b^2-e^2)}\left(a-b\mathbb{E}[c_i]+\frac{a+b\mathbb{E}[c_i]}{2b-e}\right)\\
&= \frac{a+b\mathbb{E}[c_i]}{2b-e} + \frac{b(a-(b-e)\mathbb{E}[c_i])}{(2b-e)^2}\beta\tilde{\kappa}.
\end{aligned}$$

$\square$

Proof of Proposition 8. The LHS of (16) is not impacted by any information sharing agreement. An increase in $\tilde{M}_{i\theta_j}$ decreases the RHS of this equation for all $\hat{\kappa}$, lowering the value of $\hat{\kappa}$ where the two lines intersect. An increase in $\tilde{M}_{i\rho}$ will increase the RHS for all $\hat{\kappa}$ increasing the equilibrium value of $\hat{\kappa}$. To see this second point we take the derivative of the RHS with respect to $\eta_{i\rho} = \frac{2\tilde{M}_{i\rho}}{\tilde{M}_{i\rho}+M_{i\rho}}$,

$$\begin{aligned}
\frac{\partial RHS}{\partial \eta_{i\rho}} &\overset{\text{sign}}{=} \left(1-\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)\left(-2r+2\left(r\tilde{\tilde{\tau}}_{i\rho}+\frac{1}{2}\hat{\kappa}(1-\tilde{\tilde{\tau}}_{i\rho})\right)\right)\\
&\quad +2-\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}r-\left(1-\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)\left(r\tilde{\tilde{\tau}}_{i\rho}+\frac{1}{2}\hat{\kappa}(1-\tilde{\tilde{\tau}}_{i\rho})\right).\\
&= -2r+2r\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}+2-r\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}+(2-1)\left(r\tilde{\tilde{\tau}}_{i\rho}+\frac{1}{2}\hat{\kappa}(1-\tilde{\tilde{\tau}}_{i\rho})\right)>0.
\end{aligned}$$

$\square$

D Bounds on values (for online publication)

The following inequalities are used throughout the paper.

$$\beta \in \left[0, \frac{1}{3}\right] \tag{17}$$

$$\frac{b-e}{2b-e} \in \left[0, \frac{2}{3}\right] \tag{18}$$

$$\left(\frac{b-e}{2b-e}\right)\beta \in \left[0, \frac{2}{9}\right] \tag{19}$$

$$p_\theta \in \left[\frac{1}{3}, \frac{1}{2}\right] \tag{20}$$

$$\kappa^\star \in \left[0, \frac{2}{1-\beta}\right] \subseteq [0,3] \tag{21}$$

$$\beta\kappa^\star \in \left[0, \frac{r^2}{2-r^2}\right] \subseteq \left[0, r^2\right] \subseteq [0,1] \tag{22}$$

$$\left|\frac{be}{4b^2-e^2}\right|\kappa^\star \in \left[0, \frac{r}{2-r^2}\right] \subseteq [0,r] \subseteq [0,1] \tag{23}$$

$$p_\rho \in \left[\frac{1}{9}, \frac{1}{2}\right] \text{ (when } e<0\text{)} \tag{24}$$

$$p_\rho \in [0.46, 1] \text{ (when } e>0\text{)} \tag{25}$$

D.1 Proofs of bounds

Proof of inequality (17). Since $|e| \leq b$, $\beta = e^2/(4b^2 - e^2) \geq 0$. To establish the upper bound, note that the numerator is increasing in e^2 and the denominator is decreasing in e^2, so the maximum value of β will be attained when e^2 is at its maximum. Since $e^2 \leq b^2$, it follows that $\beta \leq 3$. □

Proof of inequality (18). Since $|e| \leq b$, it is immediate that $(b-e)/(2b-e) \geq 0$. To establish the upper bound we examine the first derivative of the expression with respect to e,[43]

$$\frac{-(2b-e)+(b-e)}{(2b-e)^2} = -\frac{b}{(2b-e)^2} < 0.$$

Then the derivative is negative everywhere, and the expression is maximized when e is at its minimum, $e = -b$. This gives

$$\frac{b-(-b)}{2b-(-b)} = \frac{2}{3}.$$

[43] Basic intuition about fractions is sufficient for this maximization. We find that straightforward calculus is simpler to analyze.

$\square$

Proof of inequality (19). This follows directly from inequalities (17) and (18). $\square$

Proof of inequalities (20) *and* (21). Since $\beta\kappa^\star \geq 0$ and $p_\theta^\star = 1/(2+\beta\kappa^\star)$, it must be that $p_\theta^\star \leq 1/2$. Further, $p_\theta^\star$ will be minimized when $\beta\kappa^\star$ is maximized. Looking at $\kappa^\star$ in isolation,

$$\kappa^\star = \frac{\sigma_\theta^2 \bar{\tau}_\theta p_\theta^\star}{\sigma_\rho^2 \bar{\tau}_\rho^2 p_\rho^{\star 2} + \left(\sigma_\theta^2 + \sigma_{s\theta}^2\right) \bar{\tau}_\theta^2 p_\theta^{\star 2}}.$$

All involved terms are positive, so $\kappa^\star$ can be bounded above by assuming that $\bar{\tau}_\rho = 0$. This gives

$$\kappa^\star \leq \frac{\sigma_\theta^2 \bar{\tau}_\theta p_\theta^\star}{\left(\sigma_\theta^2 + \sigma_{s\theta}^2\right) \bar{\tau}_\theta^2 p_\theta^{\star 2}} = \frac{\sigma_\theta^2 \bar{\tau}_\theta p_\theta^\star}{\sigma_\theta^2 \bar{\tau}_\theta p_\theta^{\star 2}} = \frac{1}{p_\theta^\star}.$$

Let $\underline{p}_\theta^\star$ be the minimum feasible value of $p_\theta^\star$ and $\overline{\beta} = 1/3$ be the maximum feasible value of β; then $\kappa^\star \leq 1/\underline{p}_\theta^\star$. It follows that

$$p_\theta^\star \geq \frac{1}{2 + \frac{\beta}{\underline{p}_\theta^\star}} \quad \Longrightarrow \quad \underline{p}_\theta^\star \geq \frac{1}{2 + \frac{\overline{\beta}}{\underline{p}_\theta^\star}}.$$

This gives

$$2\underline{p}_\theta^\star + \overline{\beta} \geq 1 \quad \Longrightarrow \quad \underline{p}_\theta^\star \geq \frac{1}{3}.$$

Then $p_\theta^\star \geq 1/3$. It follows that $\kappa^\star \leq 3$. Since $|e| \leq b$, $be/(4b^2 - e^2) \leq 1/3$, hence

$$\left(\frac{be}{4b^2 - e^2}\right) \kappa^\star \leq \left(\frac{1}{3}\right) 3 = 1.$$

From $\kappa^\star \leq \frac{1}{p_\theta^\star} = 2 + \beta\kappa^\star$ we can bound

$$\kappa^\star \leq \frac{2}{1-\beta} = \frac{4 - r^2}{2 - r^2}.$$

$\square$

Proof of inequalities (22) *and* (23). This follows directly from (21),

$$0 \leq \beta\kappa^\star \leq \beta\left(\frac{2}{1-\beta}\right) = \frac{2r^2}{4 - 2r^2} = \frac{r^2}{2 - r^2}.$$

Then

$$0 \leq \left|\frac{be}{4b^2 - e^2}\right| \kappa^\star = \left|\frac{b}{c}\right| \beta\kappa^\star = \left|\frac{1}{r}\right| \beta\kappa^\star \leq \frac{r}{2 - r^2}.$$

$\square$

Proof of inequalities (24) *and* (25). Recall the equilibrium equation for $p^\star_\rho$,

$$p^\star_\rho = \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa^\star}{(2 - r\bar{\tau}_\rho) - \frac{1}{2}\left(1 - \bar{\tau}_\rho\right)\beta^2\kappa^{\star 2}}, \text{ where } r = \frac{e}{b}.$$

By inequality (22), $\beta\kappa^\star \leq |r|$, so the bound on the denominator will depend on the sign of r.

When $r < 0$, the denominator is bounded below by $2 - \beta^2\kappa^2/2$ and above by $2 - r$. The numerator is bounded above by $1 - \beta\kappa^\star/2$. This gives

$$p^\star_\rho \leq \frac{1 - \frac{1}{2}\beta\kappa^\star}{2 - \frac{1}{2}\beta^2\kappa^{\star 2}} = \frac{2 - \beta\kappa}{4 - \beta^2\kappa^{\star 2}} = \frac{1}{2 + \beta\kappa^\star} \leq \frac{1}{2};$$

$$p^\star_\rho \geq \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa^\star}{2 - r} \geq \frac{(2-r) - (1-r)}{(2-r)^2} = \frac{1}{(2-r)^2} \geq \frac{1}{9}.$$

When $r > 0$, the denominator is bounded below by $2 - r$ and above by $2 - \beta^2\kappa^{\star 2}/2$. The numerator is bounded below by $1 - \beta\kappa^\star/2$. This gives[44]

$$p^\star_\rho \leq \frac{1}{2 - r} \leq 1;$$

$$p^\star_\rho \geq \frac{1 - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{2-r^2}\right)}{2 - \frac{1}{2}\left(\frac{r^2}{4-r^2}\right)^2\kappa^{2\star}} \geq \frac{1}{2} - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{4-2r^2}\right) \gtrsim 0.46.$$

$\square$

[44]While not obvious from the approach here, simple cases and numerical investigation show that these bounds are tight.